Lecture Notes in Turbulence

Steve Berg

2nd June 2004

Contents

Chapter 1

Conservation Equations

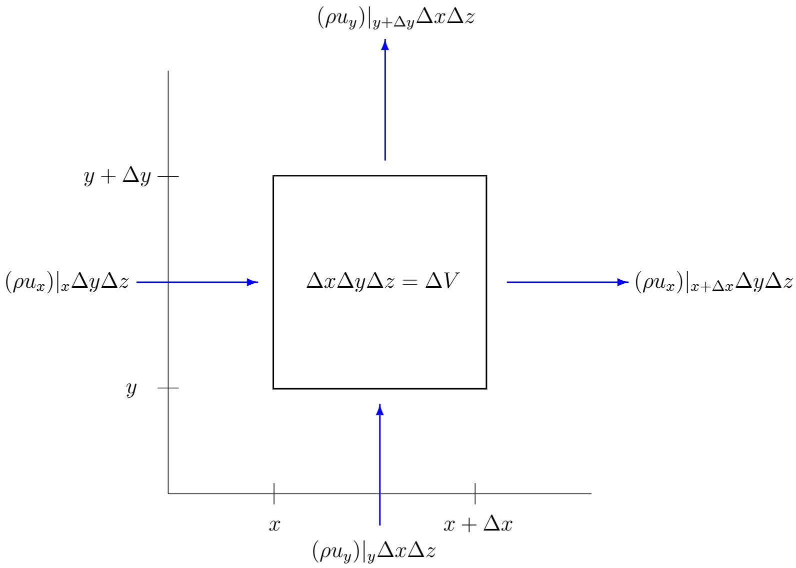
Transport equations are the basics for all CFD simulations. Before we discuss turbulence models we will derive these equations for laminar flow. Turbulence will be added in chapter 2.4. This chapter is based on lectures notes from Mathiesen (2000) and as an alternative you can also read chapter 6 in Munson et al. (2002).

1.1 Conservation of Mass

This equation is also known as equation of continuity. It is derived from the concept of control volumes, ie the mass entering a volume also has to leave it. Before we start discussing the details about mass flow, let us first define the basic equation for mass flow in one direction:

$$m_x = \rho u_x A \, [\frac{kg}{s}]$$

This equation is used for all directions for calculation of mass flow. In our case we will only derive the equation of mass for two direction (x and y). It is easier to understand the procedures for the derivation of the equations when there are a few variables as possible. The third flow direction part (z-direction) can easily be added to the two direction equation.



Accumulations of mass per time unit	=	Incoming mass per time unit	-	Outgoing mass per time unit

Accumulation of mass per time unit:

$$\frac{\partial \rho}{\partial t}\Delta V = \frac{\partial}{\partial t}(\rho u_x)\Delta x\Delta y\Delta z$$

Mass flow in and out of the control volume is then given by:

$$\underbrace{(\rho u_x)\mid_x \Delta y\Delta z}_{inflow-x} + \underbrace{(\rho u_y)\mid_y \Delta x\Delta z}_{inflow-y} - \underbrace{(\rho u_x)\mid_{x+\Delta x} \Delta y\Delta z}_{outflow-x} - \underbrace{(\rho u_y)\mid_{y+\Delta y} \Delta x\Delta z}_{outflow-y}$$

We then put the inflow and outflow for x directed flow in the same equation, and do the same for the y directed flow:

$$\left\{(\rho u_x)\mid_x - (\rho u_x)\mid_{x+\Delta x}\right\}\Delta y\Delta z$$

$$\left\{(\rho u_y)\mid_y - (\rho u_y)\mid_{y+\Delta y}\right\}\Delta x\Delta z$$

We want to express this variables with the mean value:

$$\frac{-\left\{(\rho u_x)\mid_{x+\Delta x} - (\rho u_x)\mid_x\right\}}{\Delta x}\Delta y\Delta x\Delta z$$

$$\frac{-\left\{(\rho u_y)\mid_{y+\Delta y} - (\rho u_y)\mid_y\right\}}{\Delta y}\Delta y\Delta x\Delta z$$

The only thing we have done is multiplying with respectively Δx and Δy in numerator and denominator. If we add all this parts together in one equation, the final result will be:

$$\frac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z = -\frac{\{(\rho u_x)\mid_{x+\Delta x} - (\rho u_x)\mid_x\}}{\Delta x}\Delta y\Delta x\Delta z - \frac{\{(\rho u_y)\mid_{y+\Delta y} - (\rho u_y)\mid_y\}}{\Delta y}\Delta y\Delta x\Delta$$

Δz can be removed from all parts of the equations above.

$$\frac{\partial\rho}{\partial t}\Delta x\Delta y = -\frac{\{(\rho u_x)\mid_{x+\Delta x} - (\rho u_x)\mid_x\}}{\Delta x}\Delta y\Delta x - \frac{\{(\rho u_y)\mid_{y+\Delta y} - (\rho u_y)\mid_y\}}{\Delta y}\Delta y\Delta x$$

If the control volumes becomes very small ($\Delta V \to 0$, then $\Delta x \to 0$ and $\Delta y \to 0$) , then the final equation for mass will be:

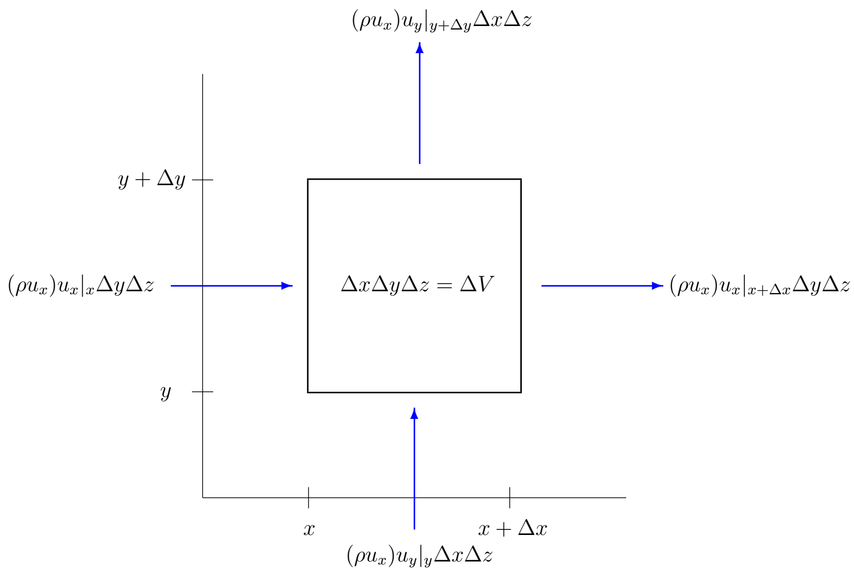
$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u_x) - \frac{\partial}{\partial y}(\rho u_y)$$

If we re-arrange this equation and add the last term for the flow in z-direction, we will end up with the final expression for mass conservation in 3 dimensions.

$$\underbrace{\frac{\partial\rho}{\partial t}}_{Accumulation} + \underbrace{\frac{\partial}{\partial x}(\rho u_x)}_{convection-x} + \underbrace{\frac{\partial}{\partial y}(\rho u_y)}_{convection-y} + \underbrace{\frac{\partial}{\partial z}(\rho u_z)}_{convection-z} = 0$$

1.2 Conservation of Moment

The equation for conservation of moment is derived from the conservation of impulse ($\rho\vec{v}$). The word impulse and moment have the same meaning and describes a force. In the deduction below we will concentrate on 2 dimensions only and the flow is x directed (ρu_x). The 3rd dimension can easily be added afterwards.



$$\boxed{\text{Accumulations of impuls per time unit}} = \boxed{\text{Incoming impuls per time unit}} - \boxed{\text{Outgoing impuls per time unit}} + \boxed{\text{Sum of all forces on the system}}$$

Accumulation of impulse per time unit can be written as:

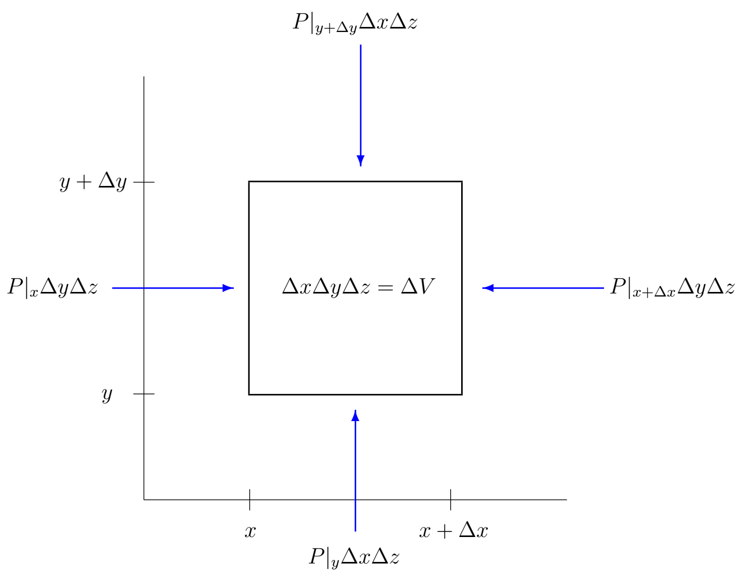
$$\frac{\partial}{\partial t}(\rho \vec{u_x})\triangle V$$

$\triangle V$ is the size of the control volume $\triangle x \triangle y \triangle z$. For the convective transport we can derive equations in the same manner as in section 1.1.

$$\underbrace{-\left(\frac{\rho u_x u_x|_{x+\Delta x} - \rho u_x u_x|_x}{\Delta x}\right)\Delta x \Delta y \Delta z}_{\text{changes of x-impuls in x-direction}} - \underbrace{\left(\frac{\rho u_x u_y|_{y+\Delta y} - \rho u_x u_y|_y}{\Delta y}\right)\Delta x \Delta y \Delta z}_{\text{changes of x-impuls in y-direction}}$$

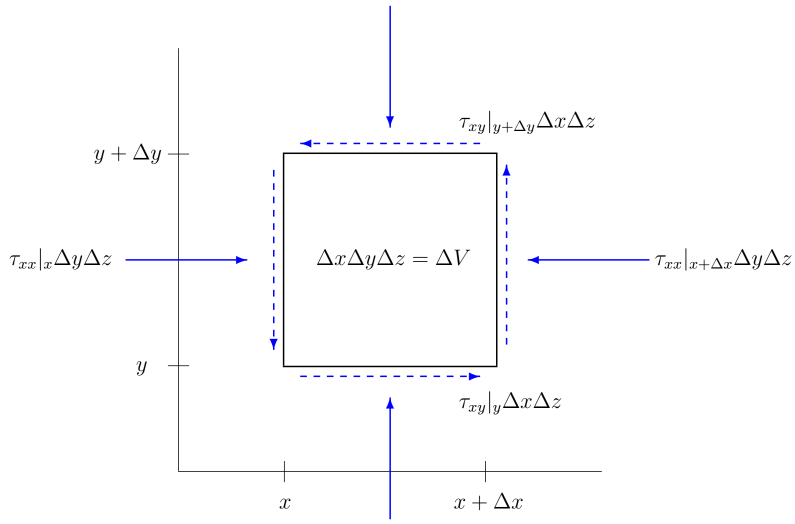
Sum of all forces on the system is a function of pressure, viscous forces and gravity.

Pressure is an isotropic variable, ie it has the same value in all directions. The conservation equation for pressure can be written as:



$$(P|_x - P|_{x+\Delta x})\Delta x \Delta z = -\left(\frac{P|_{x+\Delta x} - P|_x}{\Delta x}\right)\Delta V$$

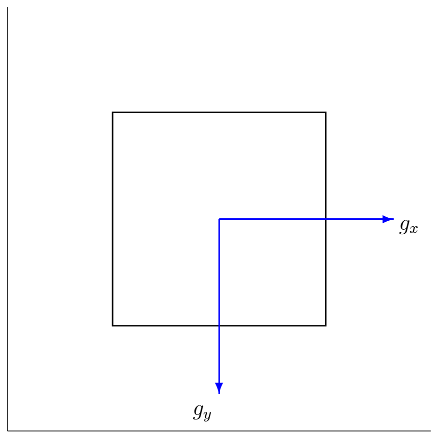
Viscous forces are diffusion of impulse caused by molecular transport (we do not include turbulence - yet). We define these stresses as shear- and normal stresses. $\tau_{xy} = \tau_{yx}$ are the shear stresses, and τ_{xx} and τ_{yy} are the normal stresses. Both have the units (N/m^2). Stresses that works in the y-direction comes in by the y-directed impulse.



Shear stress is a deformation force that alters the control volume shape while the normal stresses changes the volume.

The viscous forces are given by:

$$-\Big(\frac{\tau_{xx}|_{x+\Delta x}-\tau_{xx}|_x}{\Delta x}\Big)\Delta x\Delta y\Delta z-\Big(\frac{\tau_{xy}|_{y+\Delta y}-\tau_{xy}|_y}{\Delta x}\Big)\Delta x\Delta y\Delta z$$



Gravity forces are given by $\rho g_x\Delta V$

Now we summarise all the equations above.

$$\Delta V\frac{\partial}{\partial t}(\rho u_x)=-\Big(\frac{\rho u_x u_x|_{x+\Delta x}-\rho u_x u_x|_x}{\Delta x}\Big)\Delta V-\Big(\frac{\rho u_x u_y|_{y+\Delta y}-\rho u_x u_y|_y}{\Delta y}\Big)\Delta V$$
$$-\frac{P|_{x+\Delta x}-P|_x}{\Delta x}\Delta V-\Big(\frac{\tau_{xx}|_{x+\Delta x}-\tau_x|_x}{\Delta x}\Big)\Delta V-\Big(\frac{\tau_{xy}|_{y+\Delta y}-\tau_y|_y}{\Delta y}\Big)\Delta V+\rho g_x\Delta V$$

$\Delta V\to 0\ (\Delta x\to 0,\Delta y\to 0)$

The conservation of impulse in 2 dimensions in x-direction is:

$$\frac{\partial}{\partial t}(\rho u_x) + \frac{\partial}{\partial x}(\rho u_x u_x) + \frac{\partial}{\partial y}(\rho u_x u_y) = -\frac{\partial P}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} + \rho g_x$$

Equally, the conservation of 2 dimensional impulse in y-direction is:

$$\frac{\partial}{\partial t}(\rho u_y) + \frac{\partial}{\partial x}(\rho u_y u_x) + \frac{\partial}{\partial y}(\rho u_y u_y) = -\frac{\partial P}{\partial y} - \frac{\partial \tau_{yy}}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} + \rho g_y$$

For a 3rd dimension, an impulse in z-direction can be written in the same manner. The final equations for conservation of momentum in 3 dimensions are:

$$\frac{\partial}{\partial t}(\rho u_x)+\frac{\partial}{\partial x}(\rho u_x u_x)+\frac{\partial}{\partial y}(\rho u_x u_y)+\frac{\partial}{\partial z}(\rho u_x u_z) = -\frac{\partial P}{\partial x}-\frac{\partial \tau_{xx}}{\partial x}-\frac{\partial \tau_{xy}}{\partial y}-\frac{\partial \tau_{xz}}{\partial z}+\rho g_x$$

$$\frac{\partial}{\partial t}(\rho u_y)+\frac{\partial}{\partial x}(\rho u_y u_x)+\frac{\partial}{\partial y}(\rho u_y u_y)+\frac{\partial}{\partial z}(\rho u_y u_z) = -\frac{\partial P}{\partial x}-\frac{\partial \tau_{xx}}{\partial x}-\frac{\partial \tau_{xy}}{\partial y}-\frac{\partial \tau_{xz}}{\partial z}+\rho g_y$$

$$\frac{\partial}{\partial t}(\rho u_z)+\frac{\partial}{\partial x}(\rho u_z u_x)+\frac{\partial}{\partial y}(\rho u_z u_y)+\frac{\partial}{\partial z}(\rho u_z u_z) = -\frac{\partial P}{\partial x}-\frac{\partial \tau_{xx}}{\partial x}-\frac{\partial \tau_{xy}}{\partial y}-\frac{\partial \tau_{xz}}{\partial z}+\rho g_z$$

Written in tensor notation:

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x}(\rho u_i u_j) = -\frac{\partial P}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_j$$

Can also be written as:

$$\rho\frac{Du_j}{Dt} = -\frac{\partial P}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_i} + \rho g_i$$

The stress tensor, τ_{ij}, is fluid depended. We have already mentioned that viscosity is divided in shear- and normal stresses. For a Newtonian fluid we can write:

$$\tau_{ij} = -\mu\left(\underbrace{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}}_{\text{shear stresses}} - \underbrace{\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}}_{\text{normal stresses}}\right) \tag{1.1}$$

$$\frac{\partial u_x}{\partial x_k} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}(= 0) \tag{1.2}$$

The term k is the sum of x, y and z, and is given by equation 1.2. δ_{ij} is the Kroenicker delta and is equal to one when $i = j$, and zero when $i \neq j$. That means, when Kroenicker delta is equal to zero, normal stresses are absent, and the flow is incompressible (ρ is constant). The volume will not change, so we can ignore the last term in equation 1.1. If we include combustion, there will be large temperature gradients, and we can no longer assume that the density is constant.

The stresses in equation 1.1 can be simplified to:

$$\tau_{ij} = -\mu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$$

The derived function for τ_{ij} can be written as:

$$\frac{\partial \tau_{ij}}{\partial x_i} = -\mu\frac{\partial}{\partial x_i}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \qquad (\mu = \text{constant})$$

$$\frac{\partial \tau_{ij}}{\partial x_i} = -\mu\Big(\frac{\partial^2 u_i}{\partial x_i x_j} + \frac{\partial^2 u_i}{\partial x_i x_i}\Big)$$

$$\frac{\partial \tau_{ij}}{\partial x_i} = -\mu\Big(\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_i x_i}\Big)$$

From continuity of mass we know that for an incompressible flow:

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial \tau_{ij}}{\partial x_i} = -\mu\Big(\frac{\partial^2 u_i}{\partial x_i x_j}\Big)$$

From this we write the two general equation which are generally used in CFD: Navier-Stokes equation (1.3) and Euler equation (1.4). For Navier-Stokes we assume that both ρ and μ are constant. The Euler equation is often used in aero dynamics, since the viscosity for air is negligible ($\mu \sim 0$).

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_i \partial x_j} + \rho g_i \tag{1.3}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_i}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + \rho g_i \tag{1.4}$$

Chapter 2

Turbulent Theory

In this chapter we will explain what turbulence really is, how it is created, and how it is transferred from the mean flow to the smaller whirls. We will describe the differences between isotropic-, homogeneous- and stationary turbulence. The turbulent flow can be divided in spectrum, and we will derive equations for the smallest scales. Correlations are important in turbulence. From these numbers we can derive energy spectrum for all scales, and derive the Taylor micro scales and the Reynolds number, Re_λ, associated with it. At the end of this chapter we will derive the basic Reynolds Averaged Navier-Stokes (RANS) equation which is used in a CFD code for simulation of turbulent flow.

2.1 What is Turbulence ?

In flows which are originally laminar, turbulent arises from instabilities at large Reynolds number. In a pipe, the flow is laminar when Re $\leq$ 2100, and is turbulent when $\geq$ 4100. Between these two limits, there is an transitional zone (2100 $<$ Re $<$ 4100). This is illustrated in figure 2.1. For a flat plate the transition in boundary layers between laminar and turbulent flow occurs at Reynolds number at approximately 500 000.

The characteristic for a turbulent flow is that the fluid no longer moves as individual molecules but as macroscopic "balls" of fluid. The flow contains

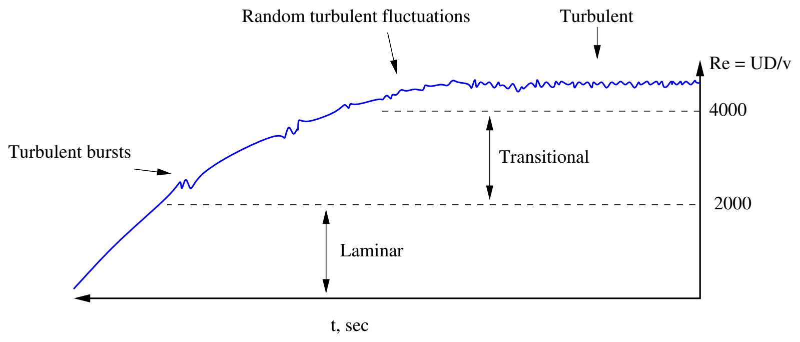


Figure 2.1: Transition from laminar to turbulent flow in a pipe

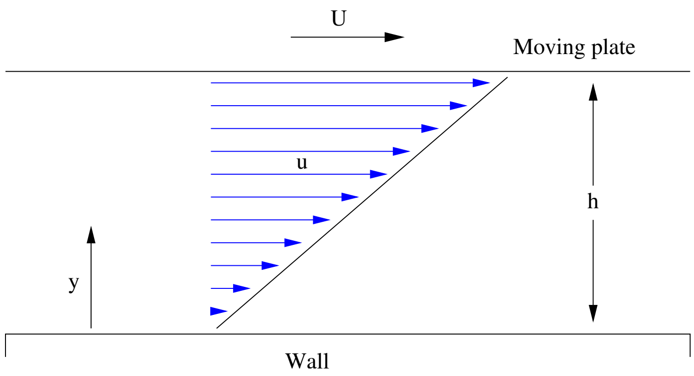


Figure 2.2: Example of Couette flow (Munson et al., 2002)

a wide range of length and time scales of the turbulent motion. The size of the largest whirls are determined by the geometry, and the size of the smallest whirls are determined by Kolmogorov scales. The flow is almost always 3 dimensional. Dissipation by viscous forces dampen turbulent unless new energy is constantly supplied. Turbulence relies on extracting energy from the mean flow, eg a shear layer. Vortex stretching is an important mechanism feeding energy into vortices.

Boussinesq assumed as early as 1877 that transport by turbulence is an extra stress, or an extra viscosity.

$$\tau_{\text{eff}} = \tau + \tau_{\text{turb}} \qquad \mu_{\text{eff}} = \mu + \mu_{\text{turb}} \tag{2.1}$$

Before we go any further, we shall discus what viscosity is. A typical example is given between two plates. One plate is at rest, and the other is moving with the flow (see figure 2.2). From experiments we know that the fluid adheres to both plates. The velocity of the lower plate is zero and the velocity at the upper plate is moving at the same velocity as the fluid. The velocity distribution between the two plates is linear. The fluid velocity is proportional to the distance y from the lower plate. In order to support the motion it is necessary to apply an tangential force to the upper plate. It is known from experiments that this force is proportional to the velocity U of the upper plate, and inversely proportional to the distance h. The frictional forces per unit area, denoted by τ, is proportional to u/h, for which in general we substitute with du/dy. The proportional factor between τ and du/dy, is often denoted by μ. And this factor depends on the nature of the fluid. It is small for thin fluids, and large for fluids like oil (viscous). The frictional forces can be written as:

$$\tau = \mu \frac{du}{dy} \tag{2.2}$$

An other word for frictional forces, τ, is shearing stress. To explain why there is a viscous force, and a velocity gradient at the boundary layer, we can imagine that there are molecules diffusing from the higher velocities in the free flow to lower velocites at the wall. And opposite, low velocity molecules are

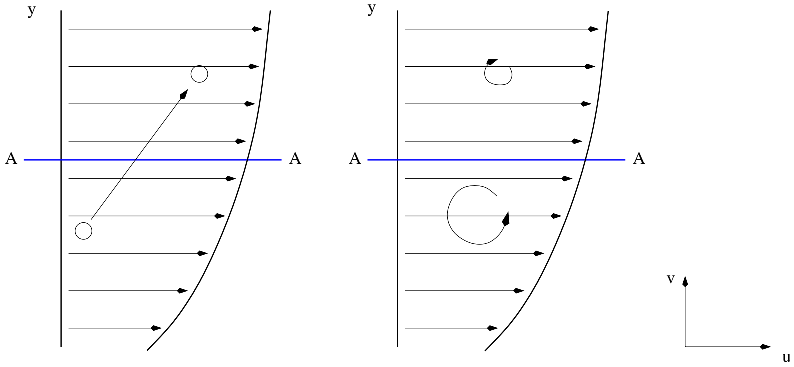


Figure 2.3: Turbulence balls is assumed to behave in the same way as molecules in laminar flow

migrating from low velocity layer to a high velocity layer. This momentum transfer will result in a shear layer, where the layer near the wall has nearly zero velocity, and the shear layers further up have an gradual increasing velocity. This idea about momentum transfer, caused by molecular movement between velocity gradients, is not restricted to boundary layers only. It also takes place in other types of flow like jets and turbulent mixing layers.

Prandtl used Boussinesq's idea in equation 2.1 to develop a simple way to calculate turbulence (algebraic or zero equation model). Prandtl used the wall mixing model where he assumed that the eddies behaves in the same way as molecules in laminar flow. Fluid lumps moving toward the low velocity regions, across line A-A, have some of their excess moment removed by the lower velocity fluid. Conversely, lumps moving away from the lower velocity region gain momentum from their new high velocity surrounding fluid. This is illustrated in figure 2.3. Lower velocity regions acts as a momentum sink. The eddies transport mass from one place to another. Flux of momentum is called stress, or said in another way, turbulent stresses is the transport of momentum ($\rho u'$) with the υ fluctuation.

$$\tau_{turb} = \upsilon'(\rho u') \tag{2.3}$$

This way of representing stress, where we assume that the eddies in turbulent flow behave in the same manner as molecules in laminar flow, is called the "eddy-viscosity approximation". We use this approximation in zero-, one- and two equations model, see section 2.4 and chapter 3. To verify that the this flux of momentum really is a stress, we can compare the units for both molecular viscosity in equation 2.2 and turbulent viscosity in equation 2.3.

$$\tau = \mu \frac{du}{dy} \left[\frac{kg}{m \cdot s} \cdot \frac{m/s}{m} = \frac{kg}{m \cdot s^2} = \frac{N}{m^2} \right]$$

$$\tau_{\text{turb}} = \rho u' v' \left[\frac{kg}{m^3} \cdot \frac{m}{s} \cdot \frac{m}{s} = \frac{kg}{m \cdot s^2} = \frac{N}{m^2} \right]$$

Turbulent Transport Larger whirls increases the transport across the flow. They transport fluids at distances equal to largest whirls, l_e. Smaller whirls does the same, but over a shorter distance. They are less important for transport, but are important for chemical reactions [1]. Larger whirls are more important for prediction of flow than smaller whirls.

Energy transfer in turbulence Energy are transferred from the mean flow to the larger whirls. This is often explained by the verse[2]:

> Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity - in the molecular sense

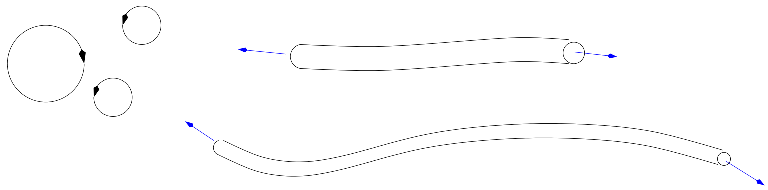


Figure 2.4: Breakdown of whirls from larger to smaller whirls (Ertesvåg, 2000)

Breakdown of turbulence (from larger to smaller whirls) takes place by two different mechanisms; larger whirls accelerate surrounding fluid to smaller whirls, and/or the whirls are stretched to longer and thinner whirls. This is illustrated in figure 2.4.

The energy is transferred to smaller whirls, until these whirls starts to disintegrate into a molecular movement. Or said in another way: the molecule transport changes from convection to diffusion where the kinetic energy is transferred into heat. This is shown in figure 2.5. This process is called dissipation. Dissipation takes place at all levels, but for high Reynolds number most of the dissipation takes place in the smaller whirls.

We notice from figure 2.5 that there can be defined a Reynolds for each individual scale. This Reynolds numbers are defined in the same way as the Reynolds number for a pipe:

$$Re_{\text{pipe}} = \frac{\rho v D}{\mu} = \frac{vD}{\upsilon}$$

v is the velocity for the fluid along the pipe and D is the pipe diameter. μ is dynamic viscosity and υ is kinematic viscosity. For a vortex, the diameter is defined as ℓ, and the vortex rotating velocity is u. Notice the difference between velocity in a pipe and in a vortex; axial velocity for a pipe and radial velocity for a vortex.

[1] We will come back to this in Combustion Technology, FACE 9

[2] From the English scientist Lewis Fry Richardson, 1922

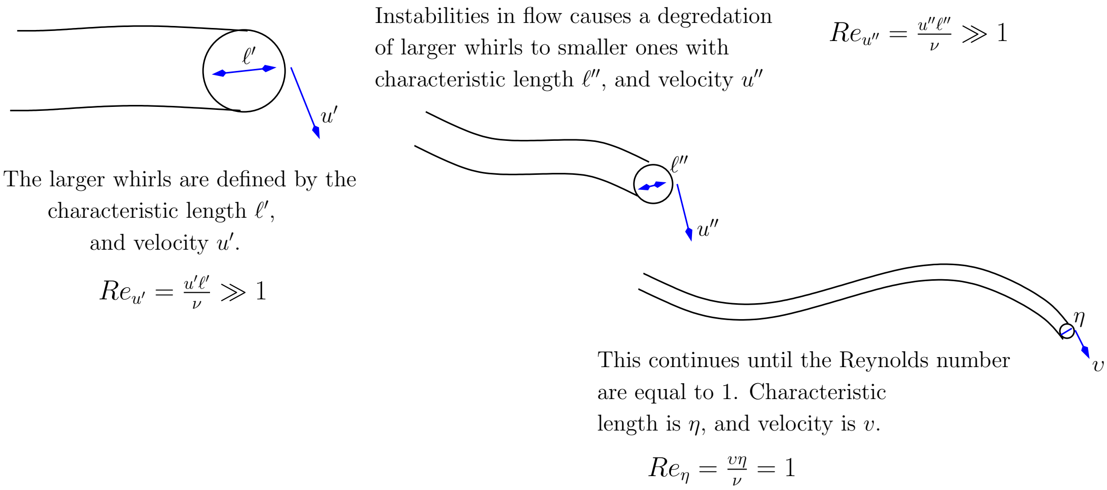


Figure 2.5: Energy is transferred from the larger to the smaller whirls (Ertesvåg, 2000)

Isotropic, Homogeneous and Stationary Turbulence Researchers have always (specially earlier times) tried to simplify turbulence. It is important to notice that these simplified turbulence hardly exist in practice. They are, however, easier to study than "real" turbulence. Some of the constants used in the turbulence models described in chapter 3, are derived under these conditions.

- In homogeneous turbulence, the fluctuating components of velocity $u'(x,t)$ and pressure $p'(x,t)$ are statistically homogeneous in the entire volume. It follows that imposed mean velocity gradients $(\partial \bar{u}_i/\partial x_j)$ must also be uniform, although they can vary with time.

  Since there are not any gradients in mean flow velocities the production of turbulence can be ignored. This will simplify the turbulence equations and they can be solved analytically.

- Isotropic turbulence is statistical independent of direction, ie the normal stresses are all equal: $\overline{u_1'^2} = \overline{u_2'^2} = \overline{u_3'^2}$. For shear stresses we got: $\overline{u_1' u_2'} = 0$ $(i \neq j)$. An mathematical explanation is that the correlations remains unchanged if the coordinate system is turned. More about correlation in section 2.3. We often assume that the smallest whirls in high Reynolds number flow is isotropic. In some special cases the larger whirls can be isotropic, eg flow behind a grid in a wind tunnel where the measuring probe is moving with the flow. Isotropic turbulence must in practice be homogeneous, ie all points in space have the same properties.

- Stationary turbulence is statistical independent of time.

2.2 Turbulence Spectrum

The main flow will transfer energy to the larger whirls. These whirls are determined by the the outer dimensions of the flow, eg the pipe diameter. Time scales for the small whirls are short. They are in close equilibrium with

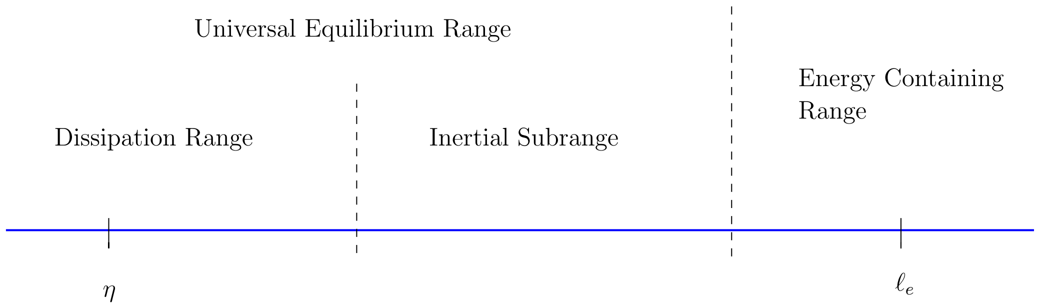


Figure 2.6: Turbulent spectrum with the larger energy containing eddies to right and dissipation eddies to left

the local properties. They are called *universal equilibrium range*. The term *universal* is used because this assumption goes for all high Reynolds number flows. The different scales in a turbulence spectrum are illustrated in figure 2.6. The spectrum is divided in 2- or 3 parts. High Reynolds number flow has 3 parts while low Reynolds number flow has only two parts. The inertial subrange vanish for lower Reynolds number. The universal equilibrium range can be divided in two separate ranges, inertial sub-range and dissipation range.

Inertial subrange In the inertial region the turbulence scales are independent of both the large scales and the small scales if the Reynolds number is large. For turbulent flow with low Reynolds number this region is “small”, ie only dissipation- and energy containing range are present.

This region is characterised by the amount of energy transported through the spectrum per time (ie ε) and the sizes of the eddy (ie $1/\kappa$). The energy in the region can be estimated as:

$$E \approx \varepsilon^a \kappa^b$$

Dimensional analysis for ε is L^2/T^3, for κ it is $1/L$, and for E it is L^3/T^2. This gives:

$$E(\kappa) = (\varepsilon^a \kappa^b) = \left(\frac{L^2}{T^3}\right)^a \left(\frac{1}{L}\right)^b = \frac{L^{2a-b}}{T^{3a}}$$

which gives: $a = \frac{2}{3}$ and $b = -\frac{5}{3}$

$$E \sim C_K \cdot \varepsilon^{\frac{2}{3}} \kappa^{-\frac{5}{3}}$$

This expression is called the Kolmogorov spectrum law or $-5/3$ power law and is illustrated in figure 2.7. This function is derived from dimensional analysis, and measurements have verified its existence. C_K is the Kolmogorov constant and is approximately 1.5.

Dissipation range Viscous forces have a greater impact on smaller whirls than the larger ones. When viscous forces overcomes the inertia movement,

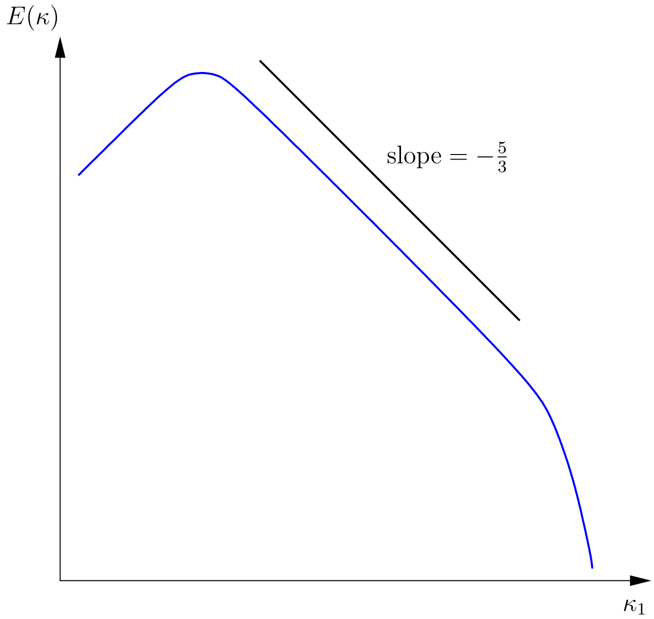


Figure 2.7: - 5/3 low for inertial subrange. Both the wavenumber κ- and the energy $E(\kappa)$ axis are given in logarithmic function

the whirls will dissipate into a molecular movement. These whirls are named after the person who first came up with this "idea", Kolmogorov. Kolmogorov also assumed that:

- small whirls are independent of larger whirls
- flow directions for the smaller whirls are independent of larger whirls - local isotropy

As illustrated in figure 2.5, Reynolds number for the smallest whirls are equal to 1, ie inertia forces equals viscous forces (or there is a balance between momentum forces and diffusion).

$$Re_\eta = \frac{\eta \upsilon}{\nu} = 1$$

The scales of the smallest eddies are determined by viscosity, ν (m^2/s), and dissipation (increase of thermal energy), ε (energy/time=m^2/s^3). The length scale, η, for the smallest eddies can be expressed as:

$$\eta = \nu^a \varepsilon^b$$

From dimensional analysis we can find that:

$$m : 1 = 2a + 2b \qquad\qquad s : 0 = -a - 3b$$

which gives $a = \frac{3}{4}$ and $b = -\frac{1}{4}$. The Kolmogorov length scale is then given by:

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}$$

Writing the velocity scales in the same way gives the Kolmogorov velocity scale υ and the Kolmogorov time scale τ as:

$$\upsilon = (\nu\varepsilon)^{\frac{1}{4}}$$

$$\tau = \left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}}$$

We have seen above that there can be derived expressions for inertial- and dissipation range. But these expressions, derived by Kolmogorov, are derived from dimensional analysis, and not from experimental measurements. There are actually smaller scales than the Kolmogorov scales. We will come back to this later in this chapter.

2.3 Correlation in Turbulence

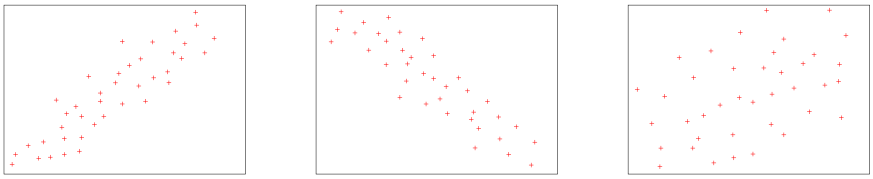
Auto correlation and cross correlation are two important statistical factors in turbulent analysis. These correlations are meant for isotropic turbulence but we also use them as approximations to real turbulence. First we need a basic explanation about what correlation really is.

Correlation is a measure of linear relationship, or the study the strength of relationship between two random variables. One simple kind of association between the variables x and y produces pairs of values or, graphically, points that scatters around a straight line. A numerical measure of this relationship is called the sample correlation coefficient (SCC), and is given in equation 2.4. Small amount of scatter around a line indicates strong association and large amount of scatter is a result of weak association. Examples are given in figure 2.8. High values for SCC indicates a strong relationship, as shown in (a). High negative values in (b) indicates that there is an anti-correlation. If values for SCC approximates 0.5, as in (c), there is not any clear relationship. A further description about correlation can be found in any basic book about basic statistics, such as Bhattacharyya and Johnson (1977).

$$SCC = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right]\left[\sum_{i=1}^{n}(y_i - \bar{y})^2)\right]}} \tag{2.4}$$

Informations about connections in time and space between fluctuations can be gathered from two-points correlations. Two-points correlations between two velocity components can be defined by equation 2.5.

$$Q_{ij} = \overline{u_i'(x)u_j'(x + \Delta x)} \tag{2.5}$$



(a) SCC=0.915 (b) SCC=-0.912 (c) SCC=0.492

Figure 2.8: Examples for different values of SCC

For standardisation we often define this function with help of the root-square-mean (σ) values at the same points. The result is:

$$R_{ij} = \frac{Q_{ij}}{\sigma_i \sigma_j}$$

The correlation coefficient is thereby limited to the interval $[-1, 1]$. In isotrop turbulence all directions possibilities are equivalent which results inn $R_{ij} = 0$ for $(i \neq j)$. R_{ij} can then be defined by two functions, $f(r)$ and $g(r)$, where r is the distance between the points $(r = \Delta x)$.

Consider two discretionary, statistic, stationary functions, u' and v', which respective mean values is zero. We introduce the term $R_{u'v'}$ for the specific two-point correlation between u' and v' (time-space correlation):

$$R_{u'v'}(x, \Delta x, \Delta t) = \frac{\overline{u'(x,t)v'(x + \Delta x, t + \Delta t)}}{\sigma_u \sigma_v}$$

We call this correlation coefficient cross-correlation; if u' and v' describe the same variable, either u' or v', we call it autocorrelation.

Auto correlation is often used in Laser Doppler Anemometry (LDA) to avoid velocity bias (Dantec, 2000). In LDA we do not measure fluid velocity directly but instead we measure seeding particles immersed in the fluid. During periods of higher velocities, a larger volume of fluid is swept through the measuring volume, and consequently a great number of velocity samples will be recorded. As a direct result, an attempt to evaluate the statistics on the flow field using arithmetic averaging will bias the result in favour of the higher velocities. One way to avoid this biasing is to find the time where two measurements are independent of each other. The autocorrelation will provide us this information. Or said in another simplified way, the autocorrelation gives us the time for a whirl to pass by the measured volume. In this kind of measurements $\Delta x = 0$, ie we are measuring in the same volume, over time.

Correlations can also be measured at two different locations. This will give us some useful information about the length scales of the eddies in turbulence. If we measure along a straight line, we can either measure the velocity fluctuations along- or by the line (longitudinal and lateral).

The longitudinal correlation is given by:

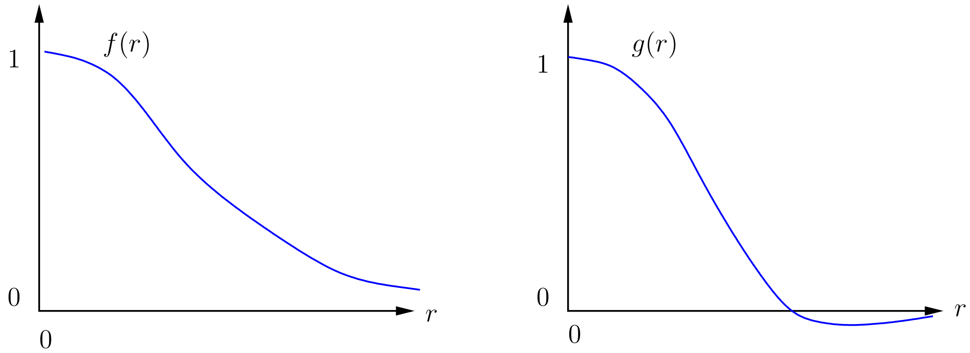
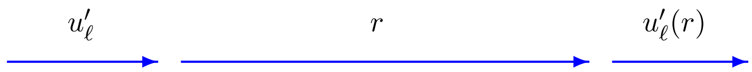
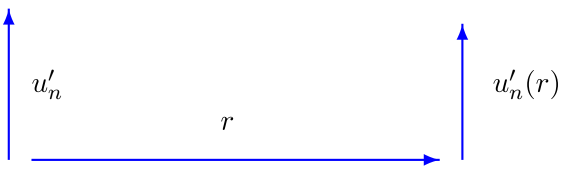


Figure 2.9: Typical examples of longitudinal- and lateral correlation



$$R_{\ell\ell}(r) = \overline{u'_\ell \cdot u'_\ell(r)} = f(r) \cdot \overline{u'^2} \qquad \Rightarrow \qquad f(r) = \frac{\overline{u'_\ell u'_\ell(r)}}{\overline{u'^2}}$$

The lateral correlation is given by:



$$R_{nn}(r) = \overline{u'_n \cdot u'_n(r)} = g(r) \cdot \overline{u'^2} \qquad \Rightarrow \qquad g(r) = \frac{\overline{u'_n u'_n(r)}}{\overline{u'^2}}$$

If two points are at the same spot, we are "looking" at the same fluctuation. The correlation is then 1.

$$f(0) = g(0) = 1$$

When distances increases, the correlation between them will gradually become smaller. If correlation is equal to zero, they are uncorrelated.

$$f(\infty) = g(\infty) = 0$$

The value for $f(r)$ can be negative for a part of of r. The value for $g(r)$ must be negative for a part of r. A fluctuation across the line must result in fluctuation in opposite direction (continuity). This is illustrated in figure 2.9

We can define a length scale L_f where the area $f(0)$ is equal to the area under the curve $f(r) \geq 0$, as indicated in figure 2.10. Since $f(0) = 1$, we can write the longitudinal macroscale:

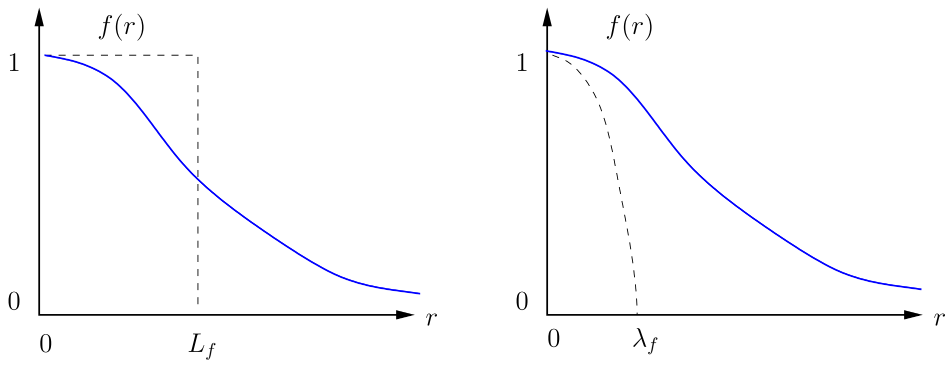


Figure 2.10: The macroscale is defined as L_f and microscale as λ_f

$$L_f = \int_0^\infty f(r)dr$$

We can define the lateral correlation function, $g(r)$, in the same way:

$$L_g = \int_0^\infty g(r)dr$$

It can be shown that $L_f = 2L_g$. These two length scales tells us the distance fluctuations will influence each other.

Taylor series are polynomials that can be used to approximate other functions. The general formula for Taylor series are given in equation 2.6.[3]

$$P_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2\ldots+\frac{f^k a}{k!}(x-a)^k+\ldots+\frac{f^n a}{n!}(x-a)^n \tag{2.6}$$

Taylor series for $f(r)$ around $r = 0$ gives:

$$f(r) = f(0) + f'(0)r + \frac{1}{2}\cdot f''(0)\cdot r^2 + \ldots$$

We know that $f(0) = 1$ and $f'(0) = 0$. If we round off the Taylor series after the r^2 link, the result will be a parable which crosses the r-axis. We define a length scale λ_f as a values instead of r where the parable crosses the r-axis ($f(r = \lambda_f) = 0$). This length scale is called the longitudinal micro scales.

$$\lambda_f^2 = -\frac{2}{f''(0)}$$

From the lateral correlation function $g(r)$ we can define a lateral microscale in the same manner as for the longitudinal micro scales.

[3]A more detailed discussion about this topic can be found in Kreyszig (1993)

$$\lambda_g^2 = -\frac{2}{g''(0)}$$

It can be shown that (Hinze, 1975) $\lambda_f = \sqrt{2}\lambda g$. Dissipation of turbulence energy is:

$$\varepsilon = v\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

For isotropic turbulence:

$$\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2} = \overline{\left(\frac{\partial u_2'}{\partial x_2}\right)^2} = \overline{\left(\frac{\partial u_3'}{\partial x_3}\right)^2}$$

$$\overline{\left(\frac{\partial u_1'}{\partial x_2}\right)^2} = \overline{\left(\frac{\partial u_1'}{\partial x_3}\right)^2} = \overline{\left(\frac{\partial u_2'}{\partial x_1}\right)^2} = \overline{\left(\frac{\partial u_2'}{\partial x_3}\right)^2} = \overline{\left(\frac{\partial u_3'}{\partial x_1}\right)^2} = \overline{\left(\frac{\partial u_3'}{\partial x_2}\right)^2}$$

$$\varepsilon = v\left[3\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2} + 6\overline{\left(\frac{\partial u_1'}{\partial x_2}\right)^2}\right]$$

According to Hinze (1975):

$$\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2} = -\overline{u'^2} \cdot f''(0) = \frac{2\overline{u'^2}}{\lambda_f^2}$$

$$\overline{\left(\frac{\partial u_1'}{\partial x_2}\right)^2} = -\overline{u'^2} \cdot 2f''(0) = \frac{4\overline{u'^2}}{\lambda_f^2} = \frac{2\overline{u'^2}}{\lambda_g^2} = -\overline{u'^2} \cdot g''(0) = 2\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2}$$

$$\varepsilon = 15v\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2} = 30v\frac{\overline{u'^2}}{\lambda_f^2} = 15v\frac{\overline{u'^2}}{\lambda_g^2}$$

It can be practical to define a length scale such as: $\lambda = \lambda_g$

$$\varepsilon = 15v\frac{u'^2}{\lambda^2} = 10v\frac{k}{\lambda^2}, \qquad u'^2 = \frac{2}{3}k$$

We only have one length scalar, as in isotropic turbulence. λ is equal to λ_g, and with this length scale we can define a turbulence number:

$$Re_\lambda = \frac{u'\lambda}{v}$$

We know that the larger whirls are determined by the outer dimensions, and the smaller whirls by viscous forces. A turbulence Reynolds number, such as Re_λ, gives a measure for the degree of turbulence. It also gives the difference between the smaller (η) and the larger (ℓ_e) whirls. If Re_λ is high, there will be a large difference between small and large whirls. It can also be shown that for isotropic turbulence (Hinze, 1975):

$$\frac{\ell_e}{\eta} \sim 0.1 Re_\lambda^{\frac{3}{2}}$$

What are Typical Values for Re_λ? Ertesvåg (2000) discusses different values for Re_λ. The highest Reynolds numbers are found in the atmosphere and oceans. The outer dimensions for these flows are huge, and typical Reynolds number are 10^4. Typical flows in laboratories are much lower. A Reynolds number (Re_λ) at 150-200 is considered high. In special cases a Reynolds number up to 10^3 is achievable.

Relationship Between Re_λ and Re_T In turbulence models with an equation for ε, there is often defined a Reynolds number:

$$Re_T = \frac{k^2}{v\varepsilon}$$

The relation between Re_λ and Re_T is according to Ertesvåg (2000):

$$u' = \left(\frac{2}{3}k\right)^{\frac{1}{2}} \qquad \lambda = \left(\frac{10vk}{\varepsilon}\right)^{\frac{1}{2}}$$

$$Re_\lambda = \frac{u'\lambda}{v} = \left(\frac{20}{3}\frac{k^2}{v\varepsilon}\right)^{\frac{1}{2}}$$

$$Re_\lambda^2 = \frac{\frac{2}{3}k \cdot 10vk}{v^2\varepsilon} = \frac{20}{3}\frac{k^2}{v\varepsilon} = \frac{20}{3}Re_T$$

Energy spectrum The two point correlation given by:

$$R_{ij}(\vec{r},t) = \overline{u_i'(\vec{x},t)u_j'(\vec{x}+\vec{r},t)}$$

can be transformed by a Fourier transformation:

$$R_{ij}(\vec{r},t) \xrightarrow{F} E_{ij}(\vec{\kappa},t)$$

E_{ij} is the energy spectrum tensor and $\vec{\kappa}$ is the wave number vector ($1/l$). The energy spectrum $E(\kappa,t)$ are independent of directions, and is defined as:

$$\int_0^\infty E(\kappa,t)d\kappa = \frac{1}{2}\overline{u_i'u_i'} = k(t)$$

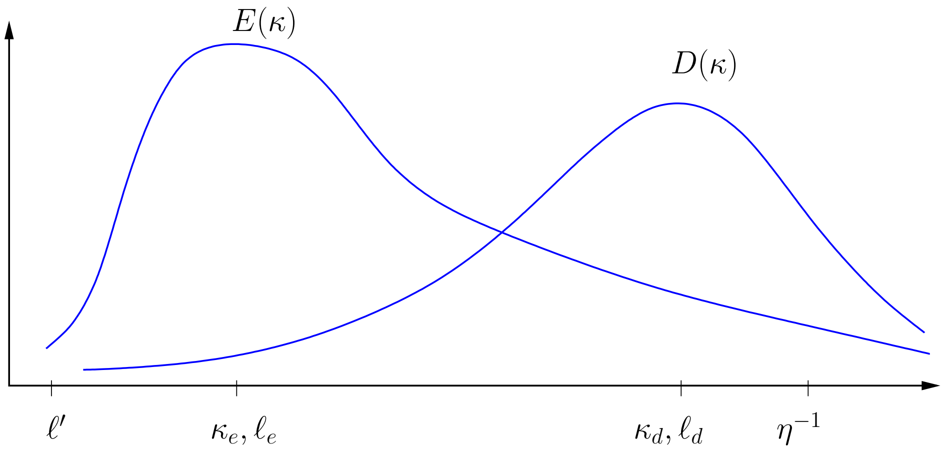


Figure 2.11: Energy and dissipation spectrum. The x-axis is logarithmic and y-axis is linear (Ertesvåg, 2000)

This energy spectrum is the area under the $E(\kappa)$ curve in figure 2.11. Dissipation spectrum $D(\kappa, t)$ is the area under the other curve in the same figure, and is defined as:

$$\int_0^\infty D(\kappa, t) d\kappa = \varepsilon(t)$$

Both curves in figure 2.11 have a maxima at different locations from the smallest (η) and largest whirls (ℓ'). The largest whirls in a flow have the same diameter as the outer dimensions, which is ℓ'. For example, the largest whirls in a pipe have the same length as the pipe diameter. There are not many of these larger whirls; most of the energy containing whirls are smaller. They are indicated in figure 2.11 by the letters ℓ_e. The larger whirls, ℓ', have more energy than the smaller whirls, ℓ_e. But ℓ_e are more numerous, so most of the energy is located around this scale. We learned earlier that the smallest whirls are the Kolmogorov scales. This scales are derived from dimensional analysis, and experiments has shown that there are actually smaller scales in a flow. Most of the dissipation, however, takes place in larger whirls (ℓ_d). Hinze (1975) is referring to some measurements which indicates that ℓ_d is typically 10 times larger than the Kolmogorov scales for isotropic flow. Pope (2000) is referring to other measurements for real flows (non-isotropic) where ℓ_d can be up to 60 times larger. Taylor scales are larger than the Kolmogorov scales, and some people often relate this scales to ℓ_d, ie that the Taylor scales represents the eddies sizes in which most of the dissipation occurs. There are some disagreement about this statement. According to Tennekes and Lumley (1972) the Taylor scales does not represent any group of eddy sizes in which dissipative effects are strong. It is not a dissipation scale, because it is defined with the assistance of a velocity scale which is not relevant for the dissipating eddies.

What Happens with Energy Spectrum when Turbulence Decreases?
When turbulence decreases, k, ε, and Re_λ has to decrease as well. But the length scales are increasing. We can imagine that when the whirls are loosing

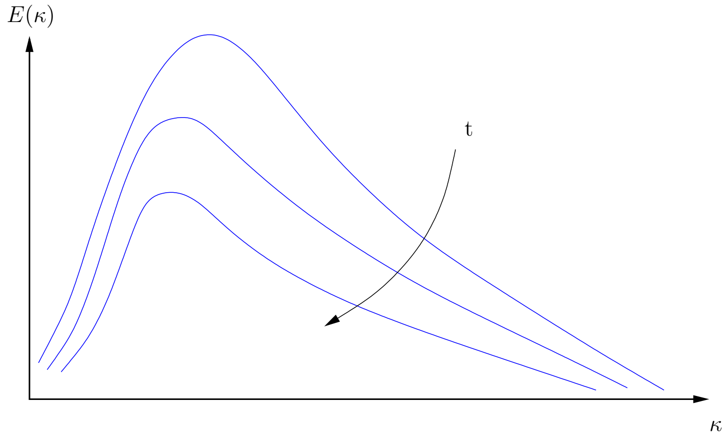


Figure 2.12: Changes in energy spectrum when turbulence decreases (Ertesvåg, 2000)

their “strength”, the particles will slip from each other and the whirls becomes larger. The area under the energy spectrum is the turbulence energy k; this area is decreasing as well. The length scales ℓ_e is increasing, and the top of the curve will move towards left (lower wave number and higher/larger length scales). Kolmogorov length scales are also increasing. The relationship ℓ_e/η (is a measure for the extent of the spectrum) is decreasing. All this are shown in figure 2.12.

The energy spectrum is decreasing and it shifts toward the longer length scales when turbulence decreases.

2.4 Turbulence Equations

We derived the Navier-Stokes equation for laminar flow in chapter 1. We repeat the equations for conservation of mass (equation 2.7) and momentum (2.8) for simplicity. We assume that $\rho =$ constant.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} = 0 \tag{2.7}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + \frac{\tau_{ij}}{\partial x_j} + \rho S_i \tag{2.8}$$

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3}\mu \cancelto{\rho = \text{const.}}{\frac{\partial u_k}{\partial x_k}} \delta_{ij}$$

Reynolds divided the velocities in a mean- and a fluctuating part. This is illustrated in figure 2.13 where the velocity in a turbulent flow is measured at a point. The instantaneous velocity is then given by equation 2.9 as the sum of mean- and fluctuating velocity.

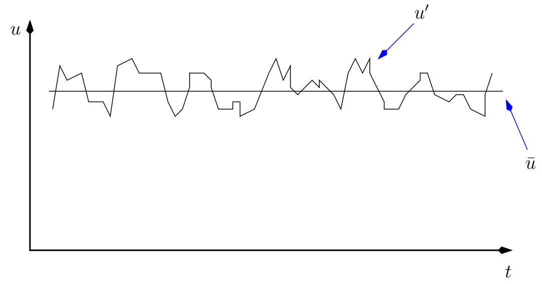


Figure 2.13: Velocities fluctuate around the mean value

$$u_i = \overline{u}_i + u'_i \tag{2.9}$$

To derive the Reynolds averaged equation, we need to insert u_i from equation 2.9 in equation 2.8.

$$\frac{\partial}{\partial t}(\rho(\overline{u}_i + u'_i)) + \frac{\partial}{\partial x_j}(\rho(\overline{u}_i + u'_i)(\overline{u}_j + u'_j)) = -\frac{\partial p}{\partial x_i} + \frac{\tau_{ij}}{\partial x_j} + \rho S_i$$

$$(\overline{u}_i + u'_i)(\overline{u}_j + u'_j) = (\overline{u_i u_j} + \overline{u}_i u'_j + u'_i \overline{u}_j + u'_i u'_j)$$

$$\frac{\partial}{\partial t}(\rho(\overline{u}_i + u'_i)) + \frac{\partial}{\partial x_j}(\rho(\overline{u_i u_j} + \overline{u}_i u'_j + u'_i \overline{u}_j + u'_i u'_j)) = -\frac{\partial p}{\partial x_i} + \frac{\tau_{ij}}{\partial x_j} + \rho S_i$$

We need to average each term in the equation. The mean value for a term with only one fluctuation is equal to zero. We need three laws for averaging:

$$\overline{\overline{\phi}} = \overline{\phi}$$

$$\overline{\phi + \psi} = \overline{\phi} + \overline{\psi}$$

$$\overline{\overline{\psi}\phi'} = \overline{\psi} \cdot \overline{\phi'}$$

$$\frac{\partial}{\partial t}(\rho(\overline{\overline{u}_i} + \cancelto{0}{\overline{u'_i}})) + \frac{\partial}{\partial x_j}(\rho(\overline{\overline{u_i u_j}} + \cancelto{0}{\overline{\overline{u}_i u'_j}} + \cancelto{0}{\overline{u'_i \overline{u}_j}} + u'_i u'_j)) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\overline{\tau}_{ij}}{\partial x_j} + \rho \overline{S}_i$$

The final result is Reynolds averaged equation (we assume that $\rho =$ constant) are given in equations 2.10 and 2.11.

$$\frac{\partial \overline{u}_j}{\partial x_j} = 0 \qquad \text{and} \qquad \frac{\partial u'_j}{\partial x_j} = 0 \tag{2.10}$$

$$\frac{\partial}{\partial t}(\rho \overline{u}_i) + \frac{\partial}{\partial x_j}(\rho \overline{u}_i \overline{u}_j) = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}(\overline{\tau}_{ij} - \rho \overline{u_i' u_j'}) + \rho \overline{S}_i \tag{2.11}$$

$\rho \overline{u_i' u_j'}$ is the Reynold stresses, and is a new unknown variable that needs to modelled. That is the purpose for a turbulence model. There are two different approaches to model the Reynold stresses, either the eddy-viscosity approach introduced in chapter 2.1, or we can model all six equations for turbulence stresses. The eddy viscosity approach is often modelled by equation 2.12. The six equation for turbulent stresses, represented by equation 2.13, can be derived from Navier-Stokes equations in the same way as we derived the expressions for the turbulent equation of momentum (equation 2.11). Definition of the different terms in equation 2.13 are discussed in chapter 3.4.

$$-\rho \overline{u_i u_j} = \mu_t \left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) - \frac{2}{3} \left(\rho k + \mu_t \cancelto{\rho = \text{const.}}{\frac{\partial \overline{u}_l}{\partial x_l}} \right) \delta_{ij} \tag{2.12}$$

$$\begin{aligned}
&\underbrace{\frac{\partial}{\partial t}\left(\rho \overline{u_i' u_j'}\right) + \frac{\partial}{\partial x_k}\left(\rho \overline{u_i' u_j'} \overline{u}_k\right)}_{\rho C_{ij}} = \underbrace{-\left(\rho \overline{u_i' u_k'} \frac{\partial \overline{u}_j}{\partial x_k} + \rho \overline{u_j' u_k'} \frac{\partial \overline{u}_i}{\partial x_k}\right)}_{\rho P_{ij}} + \underbrace{\frac{\partial}{\partial x_k}\left(\mu \frac{\partial \overline{u_i' u_j'}}{\partial x_k}\right)}_{\rho D_{ij,v}} \\
&\quad + \underbrace{\frac{\partial}{\partial x_k}\left(-\rho \overline{u_i' u_j' u_k'} - \left(\overline{p' u_i'} \delta_{jk} + \overline{p' u_j'} \delta_{ik}\right)\right)}_{\rho D_{ij,t}} + \underbrace{\overline{p' \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}}_{\rho \Phi_{ij}} \underbrace{-2\mu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}}_{\rho \varepsilon_{ij}}
\end{aligned} \tag{2.13}$$

Unfortunately, turbulence models often introduce new unknown correlations that are not known and further approximations are required.

Chapter 3

Turbulence Models

This chapter is about the different turbulence models. This are zero equation model, one equation model and the two equation models. Zero- and one equation models are (there are exceptions) incomplete, ie before we start a simulation we need to know the length scales (or time scales). Two equation models was derived in the beginning of the 70', and was a break through in turbulence modelling. The most popular two equation model, the standard k-ε model, is the most (some would probably say - "only") used turbulence model in industry.

3.1 Zero Equation Model

Zero equation model are the simplest of all turbulence model. It was developed by Prandtl in 1925. He visualised a simplified model for turbulent motion in a boundary layer where the turbulent balls moves around as molecules. He showed that the two velocity fluctuations can be given by:

$$u' = \ell \frac{d\overline{u}}{dy} \qquad\qquad v' = \ell \frac{d\overline{u}}{dy}$$

Turbulent stresses is the transport of momentum ($\rho u'$) with the v fluctuation

$$\tau_{turb} \sim v'(\rho u') \sim \rho \ell^2 \left(\frac{d\overline{u}}{dy} \right)^2$$

ℓ is a mixing length and is a function of outer dimensions, eg pipe diameter or a distance to a wall. Since ℓ is unknown, it has to be determined before any calculation starts. The zero equation model is therefore an incomplete turbulence model. This model is hardly in use any more, except some modified versions in aerodynamics. Further details are given in Wilcox (2002).

3.2 One Equation Model

In one equation models we are modelling the turbulence energy, and solving dissipation (ε) analytical. They are incomplete models since the length scales have to be related to some typical flow dimensions. This model was a fore-runner for the two equation model. There are (lately) developed complete one equation models, eg Spalart-Allmaras turbulence model.

We need to develop an equation for turbulence energy - k. Kinetic energy per mass unit in a flow is $\frac{1}{2}u_i u_i$. If we subtract the mean values from this, we end up with the kinetic energy for the turbulent fluctuations $\frac{1}{2}u'_i u'_i$. The mean value of this expression is the "mean kinetic turbulence energy" or just "turbulence energy"

$$k = \frac{1}{2}\overline{u'_i u'_i} = \frac{1}{2}(\overline{u'^2_1} + \overline{u'^2_2} + \overline{u'^2_3})$$

We will not derive the equation for k, but we will use a method from Ertesvåg (2000) and show how it can be done in 5 steps.

1. Based on equation for movement (ρu_i), and assume that ρ is constant
2. Reynolds averaging: $u_i = \overline{u}_i + u_i$
3. Subtract the last equation from the first. The result is an equation for the fluctuation $u'_i = u_i - \overline{u}_i$.

   $$\Big(\frac{\partial}{\partial t}(\rho u'_i) + \ldots = \ldots\Big) = \Big(\frac{\partial}{\partial t}(\rho u_i) + \ldots = \ldots\Big) - \Big(\frac{\partial}{\partial t}(\rho \bar{u}_i) + \ldots = \ldots\Big)$$

4. multiply this equation with fluctuation u'_i; the result is an equation for $\frac{1}{2}u'_i u'_i$ (fluctuating turbulence energy)
5. Averaging this equation will result is an expression for mean turbulence energy, $k = \frac{1}{2}\overline{u'_i u'_i}$

   $$u'_i\Big(\frac{\partial}{\partial t}(\rho u'_i) + \ldots = \ldots\Big) = \Big(\frac{\partial}{\partial t}(\rho \frac{1}{2} u'_i u'_i) + \ldots = \ldots\Big) \xrightarrow{averaging} \Big(\frac{\partial}{\partial t}(\rho \frac{1}{2}\overline{u'_i u'_i}) + \ldots = \ldots\Big)$$

The result, with constant ρ, can be written as:

$$\underbrace{\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho k \bar{u}_j)}_{\rho C_k} = \underbrace{-\rho\overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j}}_{\rho P_k} + \underbrace{\frac{\partial}{\partial x_j}\left(\mu \frac{\partial k}{\partial x_j}\right)}_{\rho D_{k,v}} + \underbrace{\frac{\partial}{\partial x_j}\Big(-\frac{1}{2}\rho\overline{u'_i u'_i u'_j} - \overline{p u'_j}\Big)}_{\rho D_{k,t}} - \underbrace{\mu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}}_{\rho\varepsilon}$$

This equation is exact, ie it has been derived from the basic Navier-stokes equation and we have not made any assumption, simplifications or trying to model any terms.

ρC_k transient- and convective term

P_k production term

$D_{k,t}$ turbulent diffusion or mean convective transport with turbulent movements

$D_{k,v}$ viscous gradient term

ε dissipation

We cannot solve the exact equation for k. Some of these terms are unknown and have to modelled (the equations is not exact any more). The triple correlation $\overline{u_i'u_i'u_j'}$ and pressure-velocity correlation $\overline{p'u_j'}$ are unknown, and $D_{k,t}$ has to be modelled. The viscous gradient term, $D_{k,v}$, can be calculated directly, and does not need any modelling. In high Reynolds number flow, this term is small compared to the turbulent viscosity, $D_{k,t}$. P_k can be calculated directly. Dissipation term, ε, has to be modelled.

Turbulent diffusion, $D_{k,t}$, is modelled as a gradient model:

$$\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j} \tag{3.1}$$

This can be interpreted in two different ways:

1. As a model for "$-\frac{1}{2}\rho\overline{u_j'u_i'u_i'} - \overline{\rho' u_j'}$"
2. As a model for "$-\frac{1}{2}\rho\overline{u_j'u_i'u_i'}$", and "$\overline{\rho' u_j'}$" is ignored

Turbulent diffusion model is:

$$\rho D_{k,t} = \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right)$$

Turbulent "balls/lumps" have a length scale L and fluctuating velocity u'. The "force" against their flow is: $F \sim \rho u'^2 \cdot A \sim \rho u'^2 \cdot L^2$ (similar to drag force). Force per time unit (effect) is: $F \cdot u' \sim \rho u'^3 \cdot L^2$. This is energy loss for one ball. If we divide by volume, $V \sim L^3$:

$$\rho\varepsilon \sim \rho\frac{u'^3}{L} \qquad \rightarrow \qquad \varepsilon \sim C_D\frac{k^{3/2}}{L}$$

C_D is a constant found from experiments. L is a characteristic length scale for the larger whirls. The expression for ε can also be found by dimensional analysis.

Molecular viscosity for a gas is given by[1] $\mu \sim \rho\ell\bar{v}$, where ℓ is a length scale and $\bar{v}$ is mean velocity for molecules. We assume that the "balls/lumps" in turbulent flows behave as molecules in gas. Turbulent viscosity can then be defined as: $\mu_t = \rho\ell' u'$

[1] From kinetic theory of gases

$$v_t \sim u'\ell' \qquad\qquad u' \sim \sqrt{2k} \sim \sqrt{k} \qquad\qquad \ell' = L$$

$$v_t = C_L\sqrt{k}L \qquad\qquad (C_L = 0.09)$$

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho k \overline{u}_j) = \frac{\partial}{\partial x_j}\left(\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) + \rho P_k - \rho\varepsilon$$

$$\rho P_k = \mu_t\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)\frac{\partial \overline{u}_i}{\partial x_j}$$

$$\mu_t = \rho v_t = \rho\sqrt{k}L$$

$$\varepsilon = C_D\frac{k^{3/2}}{L}$$

L is an algebraic expression and must be determined before the simulation starts.

3.3 Two Equation Model

As opposed to zero (analytical)- and one equation models, two equations models are complete. They can be used to predict properties of a given turbulent flow with no prior knowledge of the turbulence structures. We are using the same k - equation as for the "one equation model", and in addition we need to find a transport equation for the length- or time scale. There are several different approaches to find this extra scales, and the result is a variety of two equation models. Some of them are well documented, whilst other are not documented at all. The most popular model is the "Standard $k-\varepsilon$ model". An exact equation for ε can be derived from the Navier-Stokes equations. The final result will be:

$$\begin{aligned}\frac{\partial \varepsilon}{\partial t} + \bar{u}_j\frac{\partial \varepsilon}{\partial x_j} = &-2v\left[\overline{u'_{i,k}u'_{j,k}} + \overline{u'_{k,i}u'_{k,j}}\right]\frac{\partial \bar{u}_i}{\partial x_j} - 2v\,\overline{u'_k u'_{i,j}}\frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_j} - 2v\,\overline{u'_{i,k}u'_{i,m}u'_{k,m}}\\ &-2v^2\,\overline{u'_{i,km}u'_{i,km}} + \frac{\partial}{\partial x_j}\left[v\frac{\partial \varepsilon}{\partial x_j} - v\,\overline{u'_j u'_{i,m}u'_{i,m}} - 2\frac{v}{\rho}\overline{p'_m u'_{j,m}}\right]\end{aligned}$$

According to Wilcox (2002), this equation is far more complicated than the turbulence energy (k) equation and involves several new unknown double and triple correlations of fluctuating velocity, pressure and velocity gradients. These correlation are hopelessly difficult to measure with any degree of accuracy, so there is little hope of finding reliable guidance from experiments regarding suitable approximations.

The simplified ε equation (3.3) can be derived in a way that makes it look similar as the k equation (3.2).

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho k \bar{u}_j) = \frac{\partial}{\partial x_j}\left(\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) + \rho P_k - \rho\varepsilon \tag{3.2}$$

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \frac{\partial}{\partial x_j}(\rho \varepsilon \bar{u}_j) = \frac{\partial}{\partial x_j}\left(\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) + \rho P_\varepsilon - \rho Q_\varepsilon \tag{3.3}$$

Production and disintegration of ε is assumed to be proportional to production and disintegration of k. Each of these terms have to be multiplied with ε/k for get the right dimension.

$$P_\varepsilon = C_\varepsilon \frac{\varepsilon}{k} P_k \qquad\qquad Q_\varepsilon = C_{\varepsilon 2}\frac{\varepsilon}{k}\varepsilon$$

The idea behind this relationship is that when turbulence energy increases, the disintegration has to increase as well. Or said in another way: when P_k increases, ε should also increase, if it does not, k can get unphysical high values. The best way to increase ε is by increasing P_ε - and that is the reason why P_ε is dependent on P_k. When turbulence energy decreases, disintegration must also decrease. The complete set of equation and its constants for the standard $k - \varepsilon$ turbulence model are (Ertesvåg, 2000):

$$\mu_t = \rho v_t = C_\mu \rho \frac{k^2}{\varepsilon}$$

$$-\rho\overline{u_i' u_j'} = \mu_t\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \frac{2}{3}\rho k \delta_{ij}$$

$$\frac{\partial}{\partial t}(\rho k) + \frac{\partial}{\partial x_j}(\rho k \bar{u}_j) = \frac{\partial}{\partial x_j}\left(\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right) + \rho P_k - \rho\varepsilon$$

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \frac{\partial}{\partial x_j}(\rho \varepsilon \bar{u}_j) = \frac{\partial}{\partial x_j}\left(\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right) + C_{\varepsilon 1}\frac{\varepsilon}{k}\rho P_k - C_{\varepsilon 2}\frac{\varepsilon}{k}\rho\varepsilon$$

$$\rho P_k = \mu_t\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j}$$

$$\sigma_k = 1.0 \qquad \sigma_\varepsilon = 1.3 \qquad C_{\varepsilon 1} = 1.44 \qquad C_{\varepsilon 2} = 1.92 \qquad C_\mu = 0.09$$

Standard $k - \varepsilon$ models includes 5 universal constants. Some of these are found from boundary layer flow, C_μ and $C_{\varepsilon 1}$, others from wind tunnel experiments, $C_{\epsilon 2}$. σ_k and σ_ε are determined from computer optimisation.

What is ε ? For high Reynolds number, most of the dissipation takes place in the smaller whirls. The mechanical energy is transferred from the mean flow to the larger whirls, and thereafter to the smaller whirls.

The exact equation for ε represents the dissipating eddies, or the length scale for these. Since this equation is hard to model, we are using a simplified model where ε is given by the larger whirls. ε in the $k-\varepsilon$ model is not the dissipation, but it represents the energy transfer from larger to smaller whirls. In stationary flow the energy transfer from the larger whirls are equal to the dissipation in the smaller whirls.

Since the ε equation we use in $k-\varepsilon$ models (and RSM) differs from the exact equation derived from the Navier -Stokes equation, this equation is often seen as the weak part in the turbulence model. In standard $k-\varepsilon$ model (and other two equation models) we only have one equation for the Reynolds stresses.

Advantages with the $k-\varepsilon$ model are:

- fast (only two extra equations needs to be modelled)
- stable

Disadvantages:

- poor performance in in a variety of flows where the Reynolds stresses are not equal in all directions
  - curvatures in the flow (vortex, swirling flow, bends ...)
  - rotating flows

3.4 Reynolds Stress Model (RSM)

The Reynold stress model (RSM) does not use the eddy viscosity approach described in chapter 2.1. Instead we are deriving 6 individual equations for the stresses. This equation is already given in 2.13, but for simplicity we repeat it:

$$\underbrace{\frac{\partial}{\partial t}\left(\rho\overline{u_i'u_j'}\right)+\frac{\partial}{\partial x_k}\left(\rho\overline{u_i'u_j'}\overline{u}_k\right)}_{\rho C_{ij}}=\underbrace{-\left(\rho\overline{u_i'u_k'}\frac{\partial\overline{u}_j}{\partial x_k}+\rho\overline{u_j'u_k'}\frac{\partial\overline{u}_i}{\partial x_k}\right)}_{\rho P_{ij}}+\underbrace{\frac{\partial}{\partial x_k}\left(\mu\frac{\partial\overline{u_i'u_j'}}{\partial x_k}\right)}_{\rho D_{ij,v}}$$
$$+\underbrace{\frac{\partial}{\partial x_k}\left(-\rho\overline{u_i'u_j'u_k'}-\left(\overline{p'u_i'}\delta_{jk}+\overline{p'u_j'}\delta_{ik}\right)\right)}_{\rho D_{ij,t}}+\underbrace{\overline{p'\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}}_{\rho\Phi_{ij}}\underbrace{-2\mu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}_{\rho\varepsilon_{ij}}$$

We need an explanation what the different terms means:

C_{ij} transient term and transport with mean flow

P_{ij} production, or energy transferred from the mean flow to the Reynolds stresses

$D_{ij,v}$ viscous diffusion

$D_{ij,t}$ turbulence diffusion

Φ_{ij} re-distribution term, exchange of energy between components

ε_{ij} dissipation, transfer of kinetic energy to thermal energy

The convection term C_{ij}, production term P_{ij}, and viscous diffusion term $D_{ij,v}$ can be solved directly. The viscous diffusion term is often negligible compared to the turbulent diffusion term, but it remains important where there are significant gradients. The other terms like turbulence diffusion $D_{ij,t}$, re-distribution term Φ_{ij}, and dissipation ε_{ij} needs to be modelled.

The turbulent diffusion model is modelled as:

$$D_{ij,t} = \frac{\partial}{\partial x_k}\left(C_s\, \frac{k}{\varepsilon}\, \overline{u'_k u'_\ell}\, \frac{\partial \overline{u'_i u'_j}}{\partial x_\ell}\right) \tag{3.4}$$

C_s is often set to 0.22. k/ε is used as a time scale. $D_{ij,t}$ can be interpreted in two different ways (in the same way as in the k in equation 3.1):

1. as a model for the expression $D_{ij,t}$
2. as a model for the triple correlation gradient, and the pressure term is ignored

In practice will these two interpretations give the same expression/meaning. Another approach to model turbulent diffusion is:

$$D_{ij,t} = \frac{\partial}{\partial x_k}\left(C'_\mu\, \frac{k}{\varepsilon}\, k\, \frac{\partial \overline{u'_i u'_j}}{\partial x_k}\right) \tag{3.5}$$

This model (equation 3.5) is easier to program than the other model in equation 3.4 , and in some cases its proved to be more stable. *Fluent* uses a similar simplified approach.

Dissipation tensor, ε_{ij}, is assumed to be isotropic, ie dissipation is equal for all normal stresses.

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij}$$

Dissipation takes place in the smaller whirls. These whirls are independent of the larger whirls and the main flow. For the small whirls, the flow probability for the different directions are all equal. The dissipation must be equal distributed between the three energy components $\overline{u'^2_1}$, $\overline{u'^2_2}$ and $\overline{u'^2_3}$. This gives: $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 2/3\, \varepsilon$

The pressure strain term is the most uncertain terms we are modelling in the Reynolds stress equation. The classical approach to modelling Φ_{ij} uses the following decomposition:

$$\Phi_{ij} = \Phi_{ij,1} + \Phi_{ij,2} + \Phi_{ij,w}$$

$\Phi_{ij,1}$ slow pressure strain term, also known as return-to-isotropy term

$\Phi_{ij,2}$ rapid pressure strain term

$\Phi_{ij,w}$ wall reflection term

Pressure strain term is also called the re-distribution term. It takes from the terms who has a lot, and gives it away the the terms who has less ("Robin Hood term"). Or said in another way: it shares the energy between the components, or it makes the turbulence isotropic.

The slow pressure strain, $\Phi_{ij,1}$, is modelled as:

$$\Phi_{ij,1} = -C_1 \rho \frac{\varepsilon}{k} \left[\overline{u_i' u_j'} - \frac{2}{3} \delta_{ij} k \right]$$

where $C_1 = 1.8$. The rapid pressure strain, $\Phi_{ij,2}$, is modelled as:

$$\Phi_{ij,2} = -C_2 \left[(P_{ij} - C_{ij}) - \frac{2}{3} \delta_{ij} (P + C) \right]$$

Where $C_2 = 0.60$. $\Phi_{ij,2}$ is called the "rapid term" because C_2 is found by "rapid distortion", ie isotropic turbulence is distorted . The equation for turbulence is in these cases are simple and can be solved analytically, which gives $C_2 = 0.60$. P_{ij} and C_{ij} are taken from the Reynolds stress transport equation, $P = P_{kk}$ and $C = C_{kk}$

$\Phi_{ij,1}$ and $\Phi_{ij,2}$ re-distributes velocity fluctuations in the "free flow". When the flow approaches a wall, the velocity components will be influenced by the solid surface, and the turbulence are not seeking isotropy any more. An extra term for the re-distribution for these parts of the flow is needed.

The wall reflection term, $\Phi_{ij,w}$, is responsible for the redistribution of normal stresses near the wall. It tends to damp the normal stresses perpendicular to the wall, while enhancing the stresses parallel to the wall. Ertesvåg (1991) gives some details about near wall turbulence. All velocity fluctuation are reduced to zero at the wall ($y \to 0$), but the relationship between fluctuations and mean velocity approaches finite values. According to measurements: $u_1'/\bar{u}_1 \sim 0.25$ and $u_3'/\bar{u}_1 \sim 0.065$ when $y = 0$. The fluctuations around mean values are still intense, even when it approaches the surface. This is illustrated in figure 3.1 where the fluctuation u_1' increases in the mean flow direction $\bar{u}_1$, when the flow approaches the wall. The normal fluctuation, u_2', decreases while the u_3' fluctuations remains unchanged.

The wall reflection term, $\Phi_{ij,w}$, is given by the sum of $\Phi_{ij,1w}$ and $\Phi_{ij,2w}$:

$$\Phi_{ij,1w} = C_{1w}\frac{\varepsilon}{k}\left(\overline{u'_k u'_m} n_k n_m \delta_{ij} - \frac{3}{2}\overline{u'_i u'_k} n_j n_k - \frac{3}{2}\overline{u'_j u'_k} n_i n_k\right)\frac{k^{3/2}}{C_\ell \varepsilon d}$$

$$\Phi_{ij,2w} = C_{2w}\left(\phi_{km,2} n_k n_m \delta_{ij} - \frac{3}{2}\phi_{ik,2} n_j n_k - \frac{3}{2}\phi_{jk,2} n_i n_k\right)\frac{k^{3/2}}{C_\ell \varepsilon d}$$

where $C_{1w} = 0.5$, $C_{2w} = 0.3$. n_k is the x_k component of the unit normal to the wall, d is the normal distance to the wall, and $C_\ell = C_\mu^{3/4}/\kappa$, where $C_\mu = 0.09$ and κ is the von Karman constant ($= 0.4187$).

The wall reflection term dampens the normal fluctuations near the solid surfaces. But this term is not only a near-wall term, it also influences the flow outside the boundary layer. For flow in complex geometries, it may be difficult to define the wall distance. Developers of boundary layers and numerical techniques makes the necessary fine mesh close to the wall, and the results from the RSM are often satisfying. In industry, there are often limited time or resources to use fine grids. According to Ertesvåg (2000), there are indications that a corse grid near the wall can worsen the result compared to a k-ε model. A typical simulation which require the solution of Reynolds stresses due to swirl, rotations, etc could be solved more correctly with a k-ε model if we cannot afford a fine mesh.

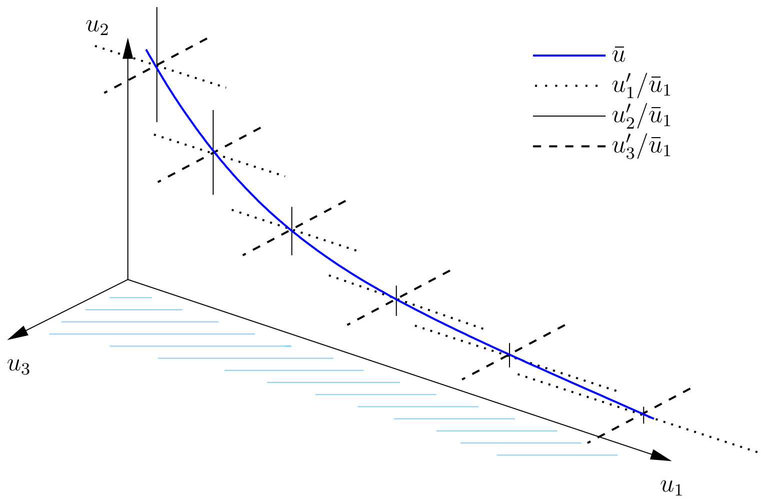


Figure 3.1: Turbulence variations for the 3 velocity components close to the wall

The advantages with RSM is that it reproduces (or trying to reproduce) the dynamics of each stress component which enables better modelling of stresses. This will lead to better capturing of (Hanjalic, 2004):

- streamline curvature (flow separation, recirculation and stagnation regions)
- strong pressure gradients
- swirling flows

- secondary motions (pressure induced and stress induced)
- three dimensionality effects
- compressibility and flow discontinuities (eg shock waves)

Chapter 4

Summary

When we use the theory for isotropic turbulence, and measure the correlation, in our case the autocorrelation, we can find the macro scales and the Taylor scales (or Taylor micro scales). The last is interesting because it gives us Re_λ. This number gives us the difference between the larger and smaller whirls, and this number must be higher than 150-200 for the existence of the inertial sub-range.

The theory for the two most interesting turbulence models, standards $k-\varepsilon$ and RSM, are discussed. The former is the most popular turbulence model and is widely used in industry. There are many other turbulence models that resemble the standard $k-\varepsilon$ model, and they might perform better in some types of flow.

The problem in turbulence modelling is to know when to use the different turbulent models. In most cases the two equation models will be adequate for many tasks. If the flow has sudden changes in flow directions, the Reynolds stress model should be used. But there are back sides with the RSM. We have already mentioned that the boundary layer should be better resolved when we are using RSM. If we cannot resolve the boundary layer, we may have to stick with two equation models, even if the physics of the flow indicates that the RSM is preferred. Another problem with the RSM are difficulties with convergence - it is harder to make a simulation converge when using RSM compared to two equation models. This may be one of the reasons why many CFD users prefer the two equations models.

Bibliography

Gouri K. Bhattacharyya and Richard A. Johnson. *Statistical Concepts and Methods.* John Wiley & sons, 1977.

Dantec. *BSA/FVA Flow Software, 4 ed.* Publication nr. 9040U5714, 2000.

Ivar S. Ertesvåg. *Utvikling av turbulensmodell for låge reynoldstal med likning for reynoldsspenningane og likning for karakteristisk frekvens.* Doktor ingeniøravhandling, Norges Tekniske Høgskole, 1991.

Ivar S. Ertesvåg. *Turbulent strøyming og forbrenning.* Tapir akademisk Forlag, Trondheim, 2000.

K. Hanjalic. Advanced turbulence closures: Second-moment- and related models. von Karman Institute for Fluid Dynamics - Lecture series in Introduction to turbulence modelling, 2004.

J. O. Hinze. *Turbulence, 2 edition.* McGraw-Hill series in mechanical engineering, 1975.

Erwin Kreyszig. *Advanced Engineering Mathematics, 7 ed.* John Wiley & Sons, Inc, 1993.

Vidar Mathiesen. Lecture notes in process modelling 1. In norwegian, 2000.

Bruce R. Munson, Donald F. Young, and Theodore H. Okiishi. *Fundamentals of Fluid Mechanics, 4 edition.* John Wiley & Sons, Inc, 2002.

Stephen B. Pope. *Turbulent Flows.* Cambridge University Press, 2000.

H. Tennekes and J. L. Lumley. *A First Course In Turbulence.* The MiT press, 1972.

David C. Wilcox. *Turbulence modelling for CFD, 2 edition.* DCW Industries, Inc, 2002.

Appendix A

Turbulence Pictures

A mixture of soap/water is flowing over pin-shaped obstacles. When the Reynolds number increases, parts of the flows becomes turbulent, and vortices are created.

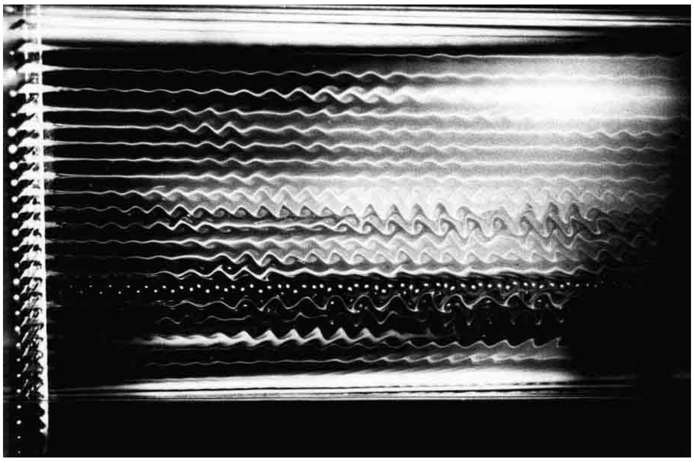


Figure A.1: Flow at low Reynolds number

Figure A.2: A von Kárman street is created below the bolts at low Reynolds number

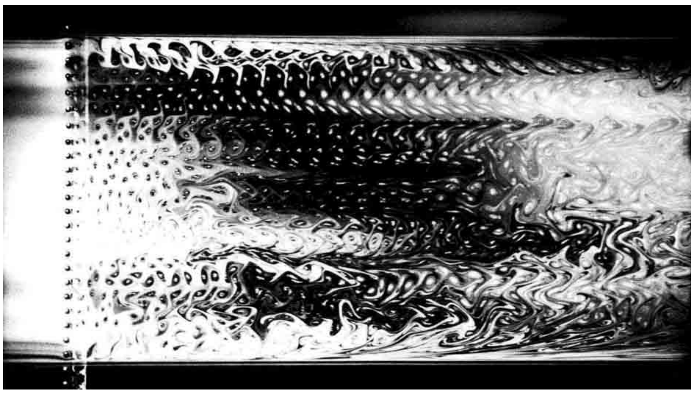


Figure A.3: Flow velocity is increased and some parts in the flow becomes turbulent

Figure A.4: Further increase in Reynolds number will make the flow turbulent, and there is a “stretching” of vortices

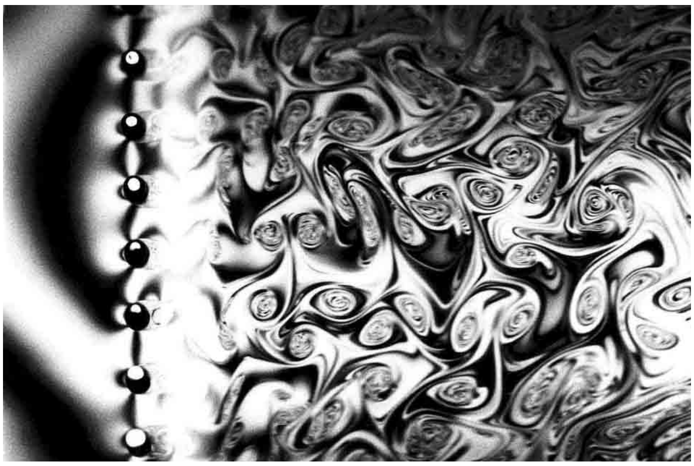


Figure A.5: Close-up of figure A.4

Appendix B

Examination Questions

Question 1: Reynolds averaging

- Describe in general terms the main features of turbulence
- Define the Reynolds averaging process
- Perform Reynolds averaging on the Navier-Stokes equations

Question 2: Size and time scales 1

- Describe in general terms the main features of turbulence
- Explain the energy cascade concept
- Define the largest and smallest scales in turbulent flow

Question 3: Size and time scales 2

- Describe in general terms the main features of turbulence
- Explain the relationship between the mean flow Reynolds number and the range of scales

Question 4: Size and time scales 3

- Describe in general terms the main features of turbulence
- Define and give examples of spatial and time correlations

Question 5: General turbulent flow

- Describe in general terms the main features of turbulence
- Define the regions of confined turbulent flow, and discuss physical aspects of these

Question 6: Reynolds averaging and closure

- Describe in general terms the main features of turbulence
- Derive the Reynolds-Averaged Navier-Stokes (RANS) equations

Question 7: Turbulence models

- Describe in general terms the main features of turbulence
- There are different turbulence models to solve the new terms, $-\overline{u_i'u_j'}$, to close the RANS equations. What is the major difference between 0-, 1-, 2-equation models and Reynolds stress models?

Question 8: Viscous stress analogy

- Describe in general terms the main features of turbulence
- Dimension analysis show that the new terms $-\rho\overline{u_i'u_j'}$ are also stresses, same as the viscous stresses, τ_{ij}. For incompressible Newtonian fluids, the viscous stresses can be expressed as:

$$\tau_{ij} = \tau_{ji} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

  In direct analogy to the viscous stresses, how do you expect to model the new terms $-\rho\overline{u_i'u_j'}$

Question 9: Boussinesq hypothesis (also known as "eddy viscosity approach")

- Describe in general terms the main features of turbulence
- Boussinesq's hypothesis reads:

$$-\rho\overline{u_i'u_j'} = \rho\nu_t\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\rho k\delta_{ij}$$

  The last term seems beyond the direct analogy to the viscous stresses. Why does the hypothesis include this term?

Question 10: The closure problem

- Describe in general terms the main features of turbulence
- By introducing Boussinesq's hypothesis, the closure problem of RANS equations turns out to be the term ν_t. What is the common basis of the mixing length (0-equation), turbulent kinetics model (1-equation) and $k-\varepsilon$ (2-equation) in solving ν_t? What are their differences?