Lectures in Turbulence for the 21st Century

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Chapter 1

The Nature of Turbulence

1.1 The turbulent world around us

The turbulent motion of fluids has captured the fancy of observers of nature for most of recorded history. From howling winds to swollen floodwaters, the omnipresence of turbulence paralyzes continents and challenges our quest for authority over the world around us. But it also delights us with its unending variety of artistic forms. Subconsciously we find ourselves observing exhaust jets on a frosty day; we are willingly hypnotized by licking flames in an open hearth. Babbling brooks and billowing clouds fascinate adult and child alike. From falling leaves to the swirls of cream in steaming coffee, turbulence constantly competes for our attention.

Turbulence by its handiwork immeasurably enriches the lives of even those who cannot comprehend its mysteries. Art museums are filled with artists attempts to depict turbulence in the world around us. The classic sketch of Italian renaissance artist and engineer, Leonardo da Vinci, shown in Figure 1.1 represents both art and early science. And as the tongue-in-cheek poem below by Corrsin (one of the turbulence greats of the past century) shows, even for those who try, the distinction between art and research is often difficult to make.

SONNET TO TURBULENCE

by

S. Corrsin[1]

(For Hans Liepmann [2] on the occasion of his 70th birthday,
with apologies to Bill S. and Liz B.B.)

Shall we compare you to a laminar flow?
You are more lovely and more sinuous.

[1]Stan Corrsin was a famous and much beloved turbulence researcher and professor at the Johns Hopkins University.

[2]Hans Liepmann was another famous turbulence researcher and professor at Cal Tech, who was Corrsin's Ph.D. dissertation advisor.

Rough winter winds shake branches free of snow,
And summer's plumes churn up in cumulus.

How do we perceive you? Let me count the ways.
A random vortex field with strain entwined.
Fractal? Big and small swirls in the maze
May give us paradigms of flows to find.

Orthonormal forms non-linearly renew
Intricate flows with many free degrees
Or, in the latest fashion, merely few —
As strange attractor. In fact, we need Cray 3's[3].

Experiment and theory, unforgiving;
For serious searcher, fun ... and it's a living!

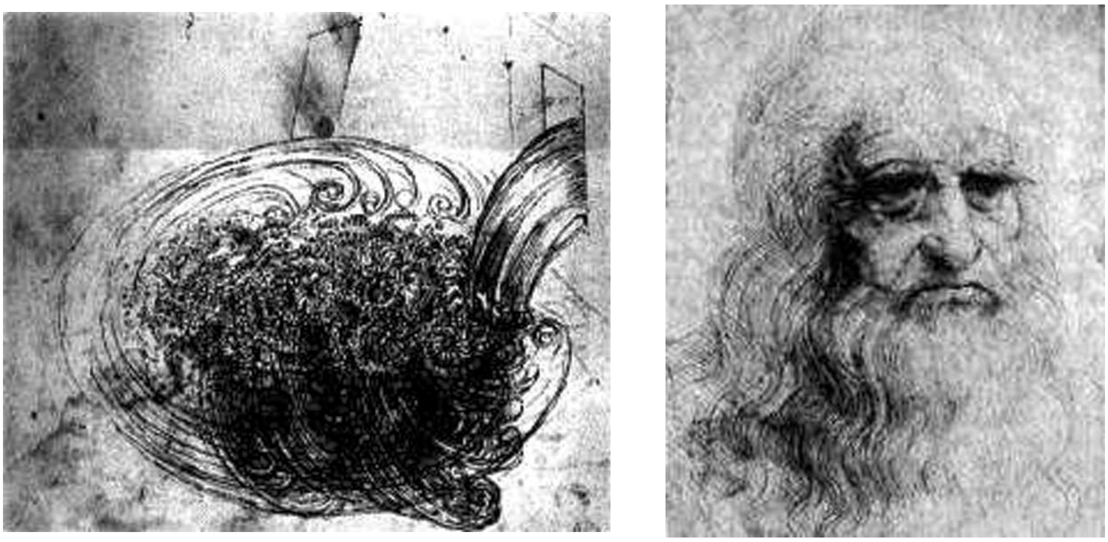

Figure 1.1: Leonardo da Vinci's observation of turbulent flow: Drawing of a free water jet issuing from a square hole into a pool (courtesy of eFluids.com).

These lectures will mostly deal with the equations used to describe the mechanics of turbulence. It is only equations which can give us the hope of predicting turbulence. But your study of this subject will be missing a great deal if this is all you learn. The advantage of studying turbulence is that you truly can see it almost everywhere as it mixes and diffuses, disrupts and dissipates the world around us.

So teach yourself to observe the natural and manmade processes around you. Not only will your life become more interesting, but your learning will be enhanced as well. Be vigilant. Whenever possible relate what you are learning to what you see. Especially note what you do not understand, and celebrate when and if you do. Then you will find that the study of turbulence really is fun.

[3]At the time this poem was written, the Cray 2 was the world's most powerful computer.

1.2 What is turbulence?

Turbulence is that state of fluid motion which is characterized by apparently random and chaotic three-dimensional vorticity. When turbulence is present, it usually dominates all other flow phenomena and results in increased energy dissipation, mixing, heat transfer, and drag. If there is no three-dimensional vorticity, there is no real turbulence. The reasons for this will become clear later; but briefly, it is ability to generate new vorticity from old vorticity that is essential to turbulence. And only in a three-dimensional flow is the necessary stretching and turning of vorticity by the flow itself possible.

For a long time scientists were not really sure in which sense turbulence is "random", but they were pretty sure it was. Like anyone who is trained in physics, we believe the flows we see around us must be the solution to some set of equations which govern. (This is after all what mechanics is about — writing equations to describe and predict the world around us.) But because of the nature of the turbulence, it wasn't clear whether the equations themselves had some hidden randomness, or just the solutions. And if the latter, was it something the equations did to them, or a consequence of the initial conditions?

All of this began to come into focus as we learned about the behavior of strongly non-linear dynamical systems in the past few decades. Even simple non-linear equations with deterministic solutions and prescribed initial conditions were found to exhibit chaotic and apparently random behavior. In fact, the whole new field of chaos was born in the 1980's[4], complete with its new language of strange attractors, fractals, and Lyapunov exponents. Such studies now play a major role in analyzing dynamical systems and control, and in engineering practice as well.

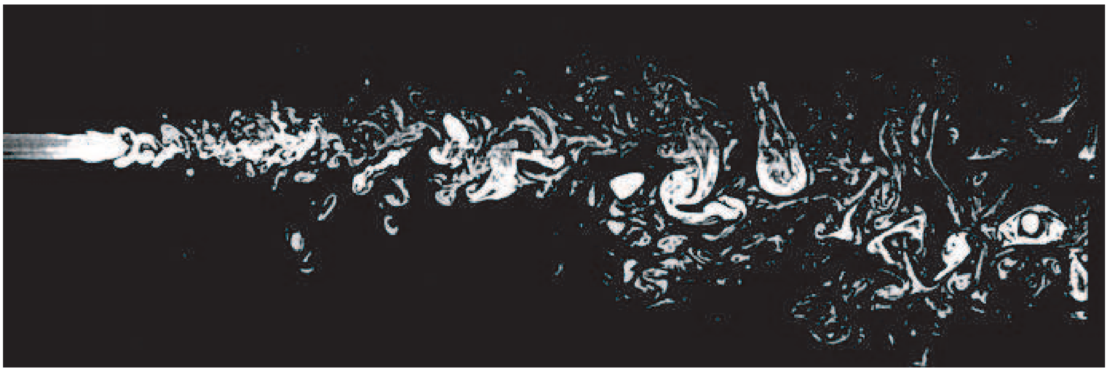

Figure 1.2: Turbulence in a water jet. Photo from Dimotakis, Miake-Lye and Papantoniou, *Phys. Flds., 26* (11), 3185 – 3192.

Turbulence is not really chaos, at least in the sense of the word that the dynamical systems people use, since turbulent flows are not only time-dependent but space dependent as well. But as even the photos of simple turbulent jets

[4]The delightful book by James Gleik "Chaos: the making of a new science" provides both interesting reading and a mostly factual account.

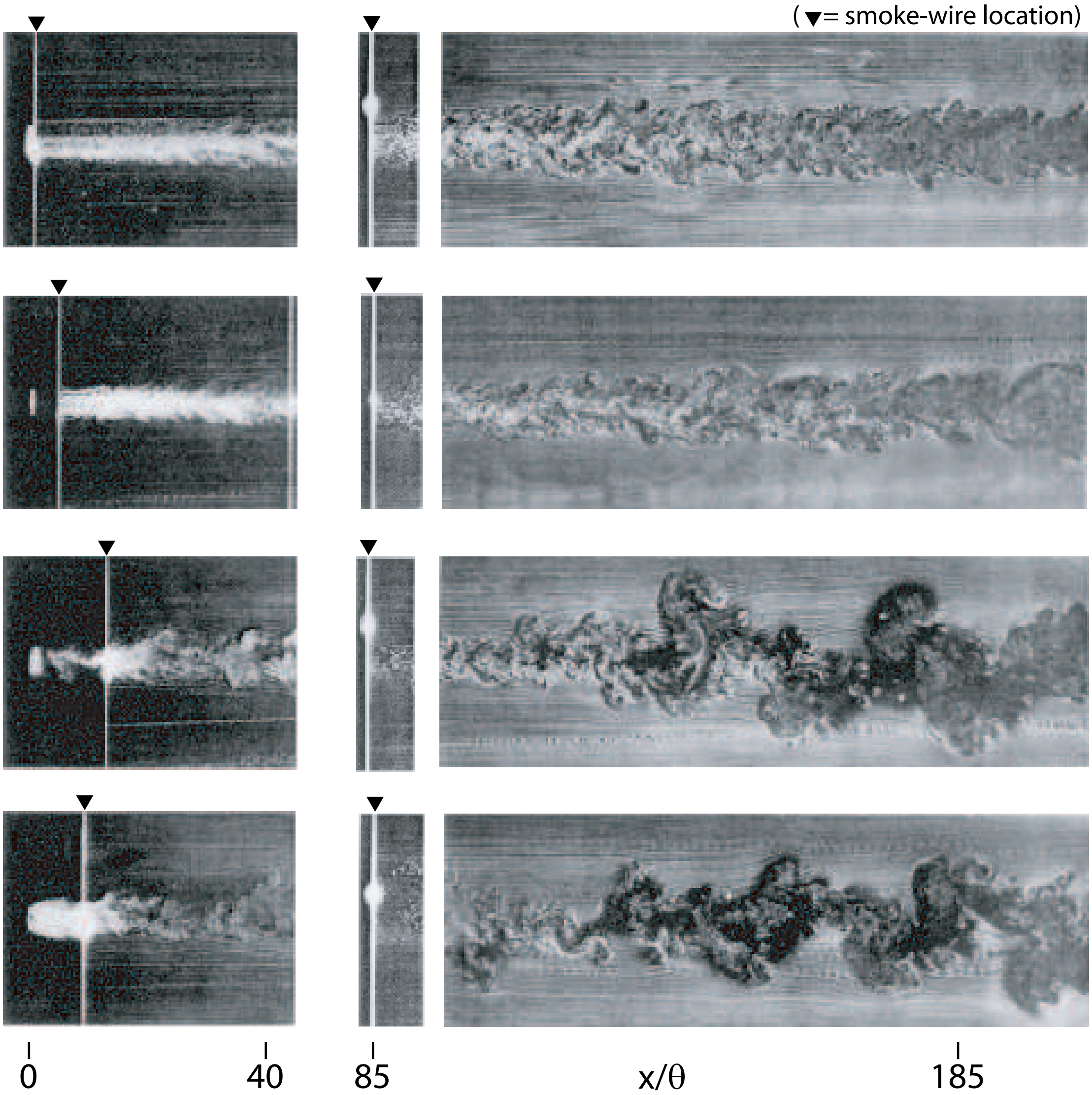

Figure 1.3: Axisymmetric wakes from four different generators. Photo from S.C. Cannon, Ph.D. Dissertation., U.of Ariz, 1991.

and wakes shown in Figures 1.2 and 1.3 make clear, turbulence has many features that closely resemble chaos. Obvious ones include spatial and temporal intermittency, dissipation, coherent structures, sensitive dependence of the instantaneous motions on the initial and upstream conditions, and even the near-fractal distribution of scales. In fact, the flows we see themselves bear an uncanny resemblance to the phase plane plots of strange attractors. No one would ever confuse a jet with a wake, but no two wakes seem to be quite alike either.

Because of the way chaos has changed our world view, most turbulence researchers now believe the solutions of the fluid mechanical equations to be deterministic. Just like the solutions of non-linear dynamical systems, we believe turbulent solutions to be determined (perhaps uniquely) by their boundary and initial conditions[5]. And like non-linear dynamical systems, these deterministic solutions of the non-linear fluid mechanics equations exhibit behavior that appears for all intents and purposes to be random. We call such solutions *turbulent*, and the phenomenon *turbulence*. Because of this chaotic-like and apparently random behavior of turbulence, we will need statistical techniques for most of our study of turbulence.

This is a course about the mechanical mysteries of turbulence. It will attempt to provide a perspective on our quest to understand it. The lack of a satisfactory understanding of turbulence presents one of the great remaining fundamental challenges to scientists — and to engineers as well, since most technologically important flows are turbulent. The advances in understanding over the past few decades, together with the advent of large scale computational and experimental capabilities, present the scientist and engineer with the first real capabilities for understanding and managing turbulent flows. As a result, this is a really wonderful time to study this subject.

1.3 Why study turbulence?

There really are the *TWO* reasons for studying turbulence — engineering and physics! And they are not necessarily complementary, *at least in the short run.*

Certainly a case can be made that we don't know enough about turbulence to even start to consider engineering problems. To begin with (as we shall see very quickly over the next few lectures), we always have fewer equations than unknowns in any attempt to predict anything other than the instantaneous motions. This is the famous **turbulence closure problem**.

Of course, closure is not a problem with the so-called DNS (Direct Numerical Simulations) in which we numerically produce the instantaneous motions in a computer using the exact equations governing the fluid. Unfortunately we won't be able to perform such simulations for real engineering problems until at least a few hundred generations of computers have come and gone. And this won't really help us too much, since even when we now perform a DNS simulation of a really simple flow, we are already overwhelmed by the amount of data and its apparently random behavior. This is because without some kind of theory, we have no criteria for selecting from it in a single lifetime what is important.

The engineer's counter argument to the scientists' lament above is:

- airplanes must fly,
- weather must be forecast,
- sewage and water management systems must be built,

[5]If it comes as a surprise to you that we don't even know this for sure, you might be even more surprised to learn that there is a million dollar prize for the person who proves it.

- society needs ever more energy-efficient hardware and gadgets.

Thus, the engineer argues, no matter the inadequate state of our knowledge, **we have the responsibility as engineers to do the best we can with what we have**. Who, considering the needs, could seriously argue with this? Almost incredibly — some physicists do!

The same argument happens in reverse as well. Engineers can become so focused on their immediate problems they too lose the big picture. The famous British aerodynamicist M. Jones captured this well when he said,

> A successful research enables problems which once seemed hopelessly complicated to be expressed so simply that we soon forget that they ever were problems. Thus the more successful a research, the more difficult does it become for those who use the result to appreciate the labour which has been put into it. This perhaps is why **the very people who live on the results of past researches are so often the most critical of the labour and effort which, in their time, is being expended to simplify the problems of the future**.

It seems evident then that there must be at least two levels of assault on turbulence. At one level, the very nature of turbulence must be explored. At the other level, our current state of knowledge — however inadequate it might be — must be stretched to provide *engineering solutions* to *real problems*.

The great danger we face is of being deceived by the successes and good fortune of our "engineering solutions" into thinking we really understand the "physics". But the real world has a way of shocking us back to reality when our "tried and tested" engineering model fails miserably on a completely new problem for which we have not calibrated it. This is what happens when we really don't understand the "physics" behind what we are doing. Hopefully this course will get you excited about both the physics and the applications, so you won't fall into this trap.

1.4 The cost of our ignorance

It is difficult to place a price tag on the cost of our limited understanding of turbulence, but it requires no imagination at all to realize that it must be enormous. Try to estimate, for example, the aggregate cost to society of our limited turbulence prediction abilities which result in inadequate weather-forecasts alone. Or try to place a value on the increased cost to the consumer of the need of the designer of virtually every fluid-thermal system —from heat exchangers to hypersonic planes— to depend on empiricism and experimentation, with the resulting need for abundant safety factors and non-optimal performance by all but the crudest measures. Or consider the frustration to engineers and cost to management of the never-ending need for "code-validation" experiments every time a new class of flows is encountered or major design change contemplated. The whole of idea of "codes" in the first place was to be able to evaluate designs without having to do experiments or build prototypes.

Some argue that our quest for knowledge about turbulence should be driven solely by the insatiable scientific curiosity of the researcher, and not by the applications. Whatever the intellectual merits of this argument, it is impossible to consider the vastness and importance of the applications and not recognize a purely financial imperative for fundamental turbulence research. The problem is, of course, that the cost of our ignorance is not confined to a single large need or to one segment of society, but is spread across the entire economic spectrum of human existence. If this were not the case, it would be easy to imagine federal involvement at the scale of America's successful moon venture or the international space station, or at very least a linear accelerator or a Galileo telescope. Such a commitment of resources would certainly advance more rapidly our understanding.

But the turbulence community — those who study and those who use the results — have failed ourselves to recognize clearly the need and nature of what we really do. Thus in turbulence, we have been forced to settle for far, far less than required to move us forward very fast, or maybe at all. Hopefully you will live to see this change. Or even better, perhaps you will be among the ones who change it.

1.5 What do we really know for sure?

Now even from these brief remarks, you have probably already figured out that the study of turbulence might be a little different than most of the courses you have taken. This is a course about a subject we are **still** studying. Now not everyone who teaches courses on this subject (and especially those who write books about it) will tell you this, but the truth is: *we really don't know a whole lot* **for sure** *about turbulence.* And worse, we even disagree about what we think we know!

Now, as you will learn in this course (or maybe heard somewhere before), there are indeed some things some researchers think we understand pretty well — like for example the Kolmogorov similarity theory for the dissipative scales and the Law of the Wall for wall-bounded flows, ideas you will soon encounter. These are based on assumptions and logical constructions about how we believe turbulence behaves in the limit of infinite Reynolds number. But even these ideas have never really been tested in controlled laboratory experiments in the limits of high Reynolds number, because no one has ever had the large scale facilities required to do so.[6]

It seems to be a characteristic of humans (and contrary to popular belief, scientists and engineers are indeed human) that we tend to accept ideas which have been around a while as *fact*, instead of just *working hypotheses* that are still waiting to be tested. One can reasonably argue that the acceptance of **most** ideas

[6]The proposal to build the Nordic Wind Tunnel at Chalmers is an attempt to fill this gap.

in turbulence is perhaps more due to the time lapsed since they were proposed and found to be in reasonable agreement with a **limited** data base, than that they have been subjected to experimental tests over the range of their assumed validity.[7] Thus it might be wise to view most 'established' laws and theories of turbulence as more like religious creeds than matters of fact.

The whole situation is a bit analogous to the old idea that the sun and stars revolved around the earth — it was a fine idea, and even good today for navigational purposes. The only problem was that one day someone (Copernicus, Brahe and Galileo among them) looked up and realized it wasn't true. So it may be with a lot of what we believe today to be true about turbulence — some day you may be the one to look at evidence in a new way and decide that things we thought to be true are wrong.

1.6 Our personal adventure

This is a *turbulence course*. You are enthused I hope, at least for the moment, to learn about turbulence. But since no two people could be in complete agreement about something like turbulence about which we know so little, this will be perhaps a pretty unusual course. I will try really hard to be honest in what I tell you. Even so, you should not trust me entirely, nor anyone else for that matter. It will really be up to you to distinguish among what you wish to consider as *fact*, *working hypothesis*, or to dismiss as *fantasy*. It is also very important that you try to keep track of which is which in your mind, and be willing to let ideas move from one category to the other as your understanding and information grows.

Like different artists painting the same scene, the pictures you and I paint will as much reflect our own personalities and histories, as the facts. But, like real works of art, both my picture of turbulence and yours might enable others to see things that they would have otherwise missed. This does *not* imply, however, that there are not real truths to be found — only that we at this point can not say with confidence what they are. Above all, we must not forget that we seek truth and understanding, the first step toward which is learning and admitting what we do not know.

Of course we will try to never completely forget that there are real problems to be solved. Throughout these lectures I will try to use many illustrations from my own experience. But the real goal is to help you develop enough fundamental understanding that you can sort through the many options available to you for the particular problems you will encounter in the real world. And maybe, with a little luck, you will even be able to make your own contribution to the state of our knowledge about turbulence. But at very least I hope the result of this course will be to make you open to new ideas, however uncomfortable they make make you feel initially.

[7] This point was made rather forcefully by Robert R. Long, (Professor Emeritus, Johns Hopkins University) in his famous footnoted Journal of Fluid Mechanics paper in 1982.

I encourage you to not be lazy. Too many study turbulence hoping for easy and quick answers, general formulas, and word pictures. The fact is the study of turbulence is quite difficult, and demands serious commitment on the part of the student. The notations are sometimes complex, and they must be this way to succinctly express the real physics. The equations themselves are extremely difficult, yet only by using them to express ideas can we say we understand the physics. Word pictures and sketches can help us, but they cannot be extrapolated to real problems. Of course we must resort to simplifications and at times even heuristic reasoning to understand what our equations are telling us. But be careful to never confuse these simplifications and pedagogical tools with the real flows you are likely to encounter. Sometimes they are useful in understanding, yet sometimes they can be misleading. There is no substitute for actually looking at a flow and analyzing exactly which terms in the governing equations are responsible for what you see.

If this all seems a bit discouraging, look at it this way. If the turbulence problem were easy, it would have been solved years ago. Like applying Newton's law (or even relativity) to point masses with known forces, every engineer could do turbulence on his laptop.[8] The turbulence problem has been worked on for over a century by many very smart people. There has certainly been progress, some would even say great progress. But not enough to make the study of turbulence easy. This problem is difficult. Even so, the equations require no more skills than undergraduate mathematics — just a lot of it. So be of brave heart and persevere. Do not quit before the end of an analysis. Actually carrying things out yourself is the only road to complete understanding. The difference between the success and failure of your effort will be almost entirely measured by your willingness to spend time and think difficult and complex thoughts. In other words, you can't be lazy and learn turbulence.

1.7 A brief outline

Now for some specifics: this course will provide an introduction to the fundamentals of turbulent flow. The focus will be on understanding the averaged equations of motion and the underlying physics they contain. The goal will be to provide you with the tools necessary to continue the study of turbulence, whether in the university or industrial setting. Topics covered include: what is turbulence; the Reynolds-averaged equations; instability and transition; simple closure models; the Reynolds stress equations; simple decaying turbulence; homogeneous shear flow turbulence; free turbulent shear flows; wall-bounded turbulent flows; multi-point and spectral considerations; and multi-point similarity in turbulence.

[8] Some indeed might now think this is possible, and for some very simple problems, it is.

Study questions for Chapter 1

1. Observe your surroundings carefully and identify at least ten different turbulent phenomena for which you can actually see flow patterns. Write down what you find particularly interesting about each.

2. Talk to people (especially engineers) you know (or even don't know particularly well) about what they think the *turbulence problem* is. Decide for yourself whether they have fallen into the trap that Professor Jones talks about in the quotation used in this text.

3. In 1990 I wrote an ASME paper entitled "The nature of turbulence". You can download a copy for yourself from the TRL website at

   www.tfd.chalmers.se/TRL — username: trl; password: trl.

   In this paper I suggested that most turbulence researchers wouldn't recognize a solution to the turbulence problem, even if they stumbled across it. My idea was that if you don't know what you are looking for, you aren't likely to know when you find it. What do you think about this, especially in light of your interviews above?

4. Some believe that computers have already (or at least soon will) make experiments in turbulence unnecessary. The simplest flow one can imagine of sufficiently high Reynolds number to really test any of the theoretical ideas about turbulence will require a computational box of approximately $(10^5)^3$, because of the large range of scales needed. The largest simulation to-date uses a computational box of $(10^3)^3$, and takes several thousand hours of processor time. Assuming computer capacity continues to double every 1.5 years, calculate how many years it will be before even this simple experiment can be done in a computer.

5. The famous aerodynamicist Theordore von Karman once said: "A scientist studies what is; an engineer creates what has never been." Think about this in the context of the comments in Chapter 1, and about the differing goals of the scientist and the engineer. Then try to figure out how you can plot a life course that will not trap you into thinking your own little corner of the world is all there is.

6. The instructor has essentially told you that you really should believe nothing he says (or anyone else says, for that matter), just because he (or they) said it. Think about what the scientific method really is, and how you will apply it to your study of the material in this course.

7. Think about the comments that ideas become accepted simply because they have been around awhile without being disproved. Can you think of examples from history, or from your own personal experience? Why do you think this happens? And how can we avoid it, at least in our work as scientists and engineers?

Chapter 2

The Elements of Statistical Analysis

2.1 Foreword

Much of the study of turbulence requires statistics and stochastic processes, simply because the instantaneous motions are too complicated to understand. This should not be taken to mean that the governing equations (usually the Navier-Stokes equations) are stochastic. Even simple non-linear equations can have deterministic solutions that look random. In other words, even though the solutions for a given set of initial and boundary conditions can be perfectly repeatable and predictable at a given time and point in space, it may be impossible to guess from the information at one point or time how it will behave at another (at least without solving the equations). Moreover, a slight change in the initial or boundary conditions may cause large changes in the solution at a given time and location; in particular, changes that we could not have anticipated.

In this chapter we shall introduce the simple idea of the *ensemble average*. Most of the statistical analyses of turbulent flows are based on the idea of an ensemble average in one form or another. In some ways this is rather inconvenient, since it will be obvious from the definitions that it is impossible to ever really measure such a quantity. Therefore we will spend the last part of this chapter talking about how the kinds of averages we can compute from data correspond to the hypothetical ensemble average we wish we could have measured. In later chapters we shall introduce more statistical concepts as we require them. But the concepts of this chapter will be all we need to begin a discussion of the averaged equations of motion in Chapter 3.

2.2 The Ensemble and Ensemble Averages

2.2.1 The mean (or ensemble) average

The concept of an *ensemble average* is based upon the existence of independent statistical events. For example, consider a number of individuals who are simultaneously flipping unbiased coins. If a value of one is assigned to a head and the value of zero to a tail, then the *arithmetic average* of the numbers generated is defined as:

$$X_N = \frac{1}{N}\Sigma x_n \tag{2.1}$$

where our nth flip is denoted as x_n and N is the total number of flips.

Now if all the coins are the same, it doesn't really matter whether we flip one coin N times, or N coins a single time. The key is that they must all be *independent events* — meaning the probability of achieving a head or tail in a given flip must be completely independent of what happens in all the other flips. Obviously we can't just flip one coin once and count it N times; these clearly would not be independent events.

Exercise Carry out an experiment where you flip a coin 100 times in groups of 10 flips each. Compare the values you get for X_{10} for each of the 10 groups, and note how they differ from the value of X_{100}.

Unless you had a very unusual experimental result, you probably noticed that the value of the X_{10}'s was also a random variable and differed from ensemble to ensemble. Also the greater the number of flips in the ensemble, the closer you got to $X_N = 1/2$. Obviously the bigger N, the less fluctuation there is in X_N.

Now imagine that we are trying to establish the nature of a random variable, x. The nth *realization* of x is denoted as x_n. The *ensemble average* of x is denoted as X (or $< x >$), and *is defined as*

$$X =< x >\equiv \frac{1}{N}\lim_{N\to\infty}\sum_{n=1}^{N} x_n \tag{2.2}$$

Obviously it is impossible to obtain the ensemble average experimentally, since we can never have an infinite number of independent realizations. The most we can ever obtain is the arithmetic mean for the number of realizations we have. For this reason the arithmetic mean can also referred to as the *estimator* for the true mean or ensemble average.

Even though the true mean (or ensemble average) is unobtainable, nonetheless, the idea is still very useful. Most importantly, we can almost always be sure the ensemble average exists, even if we can only estimate what it really is. The fact of its existence, however, does not always mean that it is easy to obtain in practice. All the theoretical deductions in this course will use this ensemble average. Obviously this will mean we have to account for these "statistical differences" between true means and estimates of means when comparing our theoretical results to actual measurements or computations.

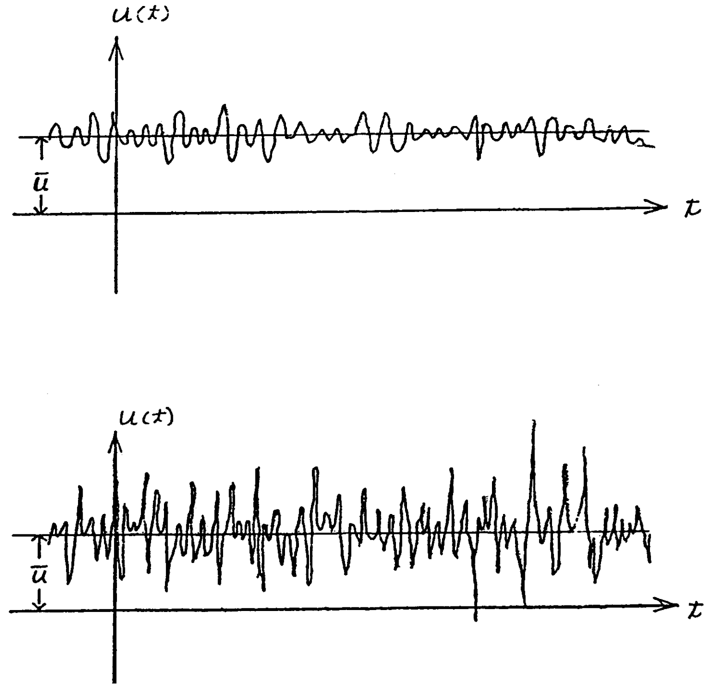

Figure 2.1:

In general, the x_n could be realizations of any random variable. The X defined by equation 2.2 represents the ensemble average of it. The quantity X is sometimes referred to as the *expected value* of the random variable x, or even simply its *mean.*

For example, the velocity vector at a given point in space and time, $\vec{x}, t$, in a given turbulent flow can be considered to be a random variable, say $u_i(\vec{x}, t)$. If there were a large number of identical experiments so that the $u_i^{(n)}(\vec{x}, t)$ in each of them were identically distributed, then the ensemble average of $u_i^{(n)}(\vec{x}, t)$ would be given by

$$< u_i(\vec{x}, t) >= U_i(\vec{x}, t) \equiv \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} u_i^{(n)}(\vec{x}, t) \tag{2.3}$$

Note that this ensemble average, $U_i(\vec{x}, t)$, will, in general, vary with the independent variables $\vec{x}$ and t. It will be seen later that under certain conditions the ensemble average is the same as the average which would be generated by averaging in time. Even when a time average is not meaningful, however, the ensemble average can still be defined; e.g., as in an non-stationary or periodic flow. Only ensemble averages will be used in the development of the turbulence equations in this book unless otherwise stated.

2.2.2 Fluctuations about the mean

It is often important to know how a random variable is distributed about the mean. For example, Figure 2.1 illustrates portions of two random functions of time which have identical means, but are obviously members of different ensembles since the amplitudes of their fluctuations are not distributed the same. It is possible to distinguish between them by examining the statistical properties of the fluctuations about the mean (or simply the fluctuations) defined by:

$$x' = x - X \tag{2.4}$$

It is easy to see that the average of the fluctuation is zero, i.e.,

$$< x' >= 0 \tag{2.5}$$

On the other hand, the ensemble average of the square of the fluctuation is *not* zero. In fact, it is such an important statistical measure we give it a special name, the **variance**, and represent it symbolically by either $var[x]$ or $< (x')^2 >$. The *variance* is defined as:

$$var[x] \equiv < (x')^2 >=< [x - X]^2 > \tag{2.6}$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} [x_n - X]^2 \tag{2.7}$$

Note that the variance, like the ensemble average itself, can never really be measured, since it would require an infinite number of members of the ensemble.

It is straightforward to show from equation 2.2 that the variance in equation 2.6 can be written as:

$$var[x] =< x^2 > -X^2 \tag{2.8}$$

Thus the variance is the *second-moment* minus the square of the *first-moment* (or mean). In this naming convention, the ensemble mean is the *first moment.*

Exercise Use the definitions of equations 2.2 and 2.7 to derive equation 2.8.

The variance can also referred to as the *second central moment of x.* The word central implies that the mean has been subtracted off before squaring and averaging. The reasons for this will be clear below. If two random variables are identically distributed, then they must have the same mean and variance.

The variance is closely related to another statistical quantity called the *standard deviation* or root mean square (*rms*) value of the random variable x,which is denoted by the symbol, σ_x. Thus,

$$\sigma_x \equiv (var[x])^{1/2} \tag{2.9}$$

or $\sigma_x^2 = var[x]$.

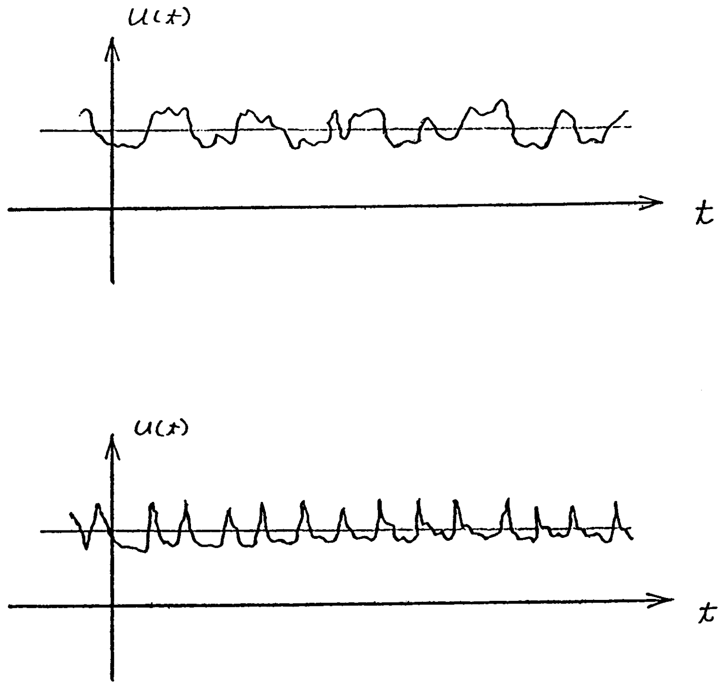

Figure 2.2:

2.2.3 Higher moments

Figure 2.2 illustrates two random variables of time which have the same mean and also the same variances, but clearly they are still quite different. It is useful, therefore, to define higher moments of the distribution to assist in distinguishing these differences.

The m-th moment of the random variable is defined as:

$$< x^m >= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} x_n^m \tag{2.10}$$

It is usually more convenient to work with the *central moments* defined by:

$$< (x')^m >=< (x - X)^m >= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} [x_n - X]^m \tag{2.11}$$

The central moments give direct information on the distribution of the values of the random variable about the mean. It is easy to see that the variance is the second central moment (i.e., $m = 2$).

2.3 Probability

2.3.1 The histogram and probability density function

The frequency of occurrence of a given *amplitude* (or value) from a finite number of realizations of a random variable can be displayed by dividing the range of possible values of the random variables into a number of slots (or windows). Since all possible values are covered, each realization fits into only one window. For every realization a count is entered into the appropriate window. When all the realizations have been considered, the number of counts in each window is divided by the total number of realizations. The result is called the **histogram** (or *frequency of occurrence* diagram). From the definition it follows immediately that the sum of the values of all the windows is exactly one.

The shape of a histogram depends on the *statistical distribution of the random variable*, but it also depends on the total number of realizations, N, and the size of the slots, Δc. The histogram can be represented symbolically by the function $H_x(c, \Delta c, N)$ where $c \leq x < c + \Delta c$, Δc is the slot width, and N is the number of realizations of the random variable. Thus the histogram shows the relative frequency of occurrence of a given value range in a given ensemble. Figure 2.3 illustrates a typical histogram. If the size of the sample is increased so that the number of realizations in each window increases, the diagram will become less erratic and will be more representative of the actual *probability* of occurrence of the amplitudes of the signal itself, as long as the window size is sufficiently small.

If the number of realizations, N, increases without bound as the window size, Δc, goes to zero, the histogram divided by the window size goes to a limiting curve called the probability density function, $B_x(c)$. That is,

$$B_x(c) \equiv \lim_{\substack{N \to \infty \\ \Delta c \to 0}} H(c, \Delta c, N)/\Delta c \tag{2.12}$$

Note that as the window width goes to zero, so does the number of realizations which fall into it, NH. Thus it is only when this number (or relative number) is divided by the slot width that a meaningful limit is achieved.

The **probability density function** (or **pdf**) has the following properties:

- Property 1:
$$B_x(c) > 0 \tag{2.13}$$
always.

- Property 2:
$$Prob\{c < x < c + dc\} = B_x(c)dc \tag{2.14}$$
where $Prob\{\ \}$ is read "the probability that".

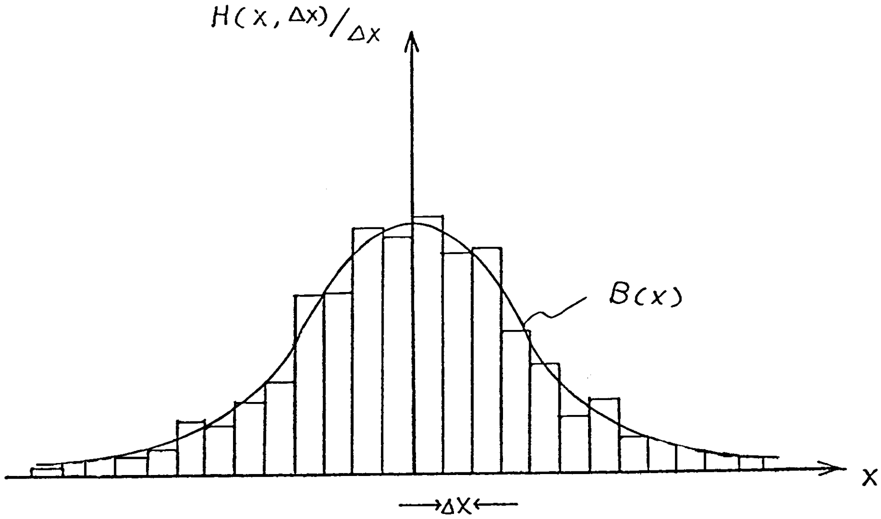

Figure 2.3:

- Property 3:

$$Prob\{x < c\} = \int_{-\infty}^{x} B_x(c)dc \tag{2.15}$$

- Property 4:

$$\int_{-\infty}^{\infty} B_x(x)dx = 1 \tag{2.16}$$

The condition imposed by property (1) simply states that negative probabilities are impossible, while property (4) assures that the probability is unity that a realization takes on some value. Property (2) gives the probability of finding the realization in a interval around a certain value, while property (3) provides the probability that the realization is less than a prescribed value. Note the necessity of distinguishing between the running variable, x, and the integration variable, c, in equations 2.14 and 2.15.

Since $B_x(c)dc$ gives the probability of the random variable x assuming a value between c and $c + dc$, any moment of the distribution can be computed by integrating the appropriate power of x over all possible values. Thus the n-th moment is given by:

$$< x^n >= \int_{-\infty}^{\infty} c^n B_x(c)dc \tag{2.17}$$

Exercise: Show (by returning to the definitions) that the value of the moment determined in this manner is exactly equal to the ensemble average defined earlier in equation 2.10. (Hint: use the definition of an integral as a limiting sum.)

If the probability density is given, the moments of all orders can be determined. For example, the variance can be determined by:

$$var\{x\} = <(x-X)^2> = \int_{-\infty}^{\infty} (c-X)^2 B_x(c) dc \tag{2.18}$$

The central moments give information about the shape of the probability density function, and *vice versa.* Figure 2.4 shows three distributions which have the same mean and standard deviation, but are clearly quite different. Beneath them are shown random functions of time which might have generated them. Distribution (b) has a higher value of the fourth central moment than does distribution (a). This can be easily seen from the definition

$$<(x-X)^4> = \int_{-\infty}^{\infty} (c-X)^4 B_x(c) dc \tag{2.19}$$

since the fourth power emphasizes the fact that distribution (b) has more weight in the tails than does distribution (a).

It is also easy to see that because of the symmetry of pdf's in (a) and (b), all the odd central moments will be zero. Distributions (c) and (d), on the other hand, have non-zero values for the odd moments, because of their asymmetry. For example,

$$<(x-X)^3> = \int_{-\infty}^{\infty} (c-X)^3 B_x(c) dc \tag{2.20}$$

is equal to zero if B_x is an even function.

2.3.2 The probability distribution

Sometimes it is convenient to work with the **probability distribution** instead of with the probability density function. The probability distribution is defined as the probability that the random variable has a value less than or equal to a given value. Thus from equation 2.15, the probability distribution is given by

$$F_x(c) = Prob\{x < c\} = \int_{-\infty}^{c} B_x(c') dc' \tag{2.21}$$

Note that we had to introduce the integration variable, c', since c occurred in the limits.

Equation 2.21 can be inverted by differentiating by c to obtain

$$B_x(c) = \frac{dF_x}{dc} \tag{2.22}$$

2.3.3 Gaussian (or normal) distributions

One of the most important pdf's in turbulence is the Gaussian or Normal distribution defined by

$$B_{xG}(c) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-(c-X)^2/2\sigma^2} \tag{2.23}$$

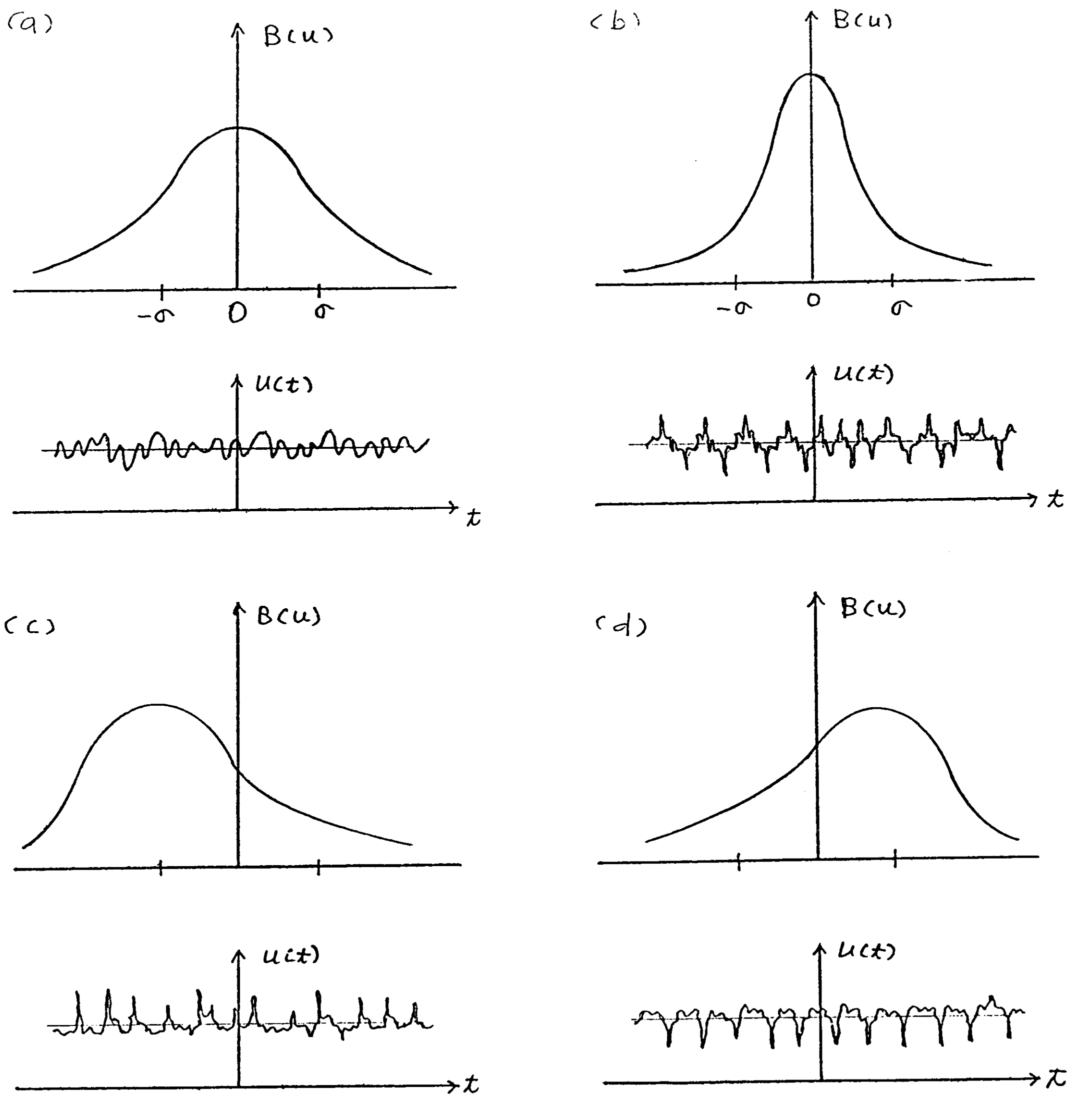

Figure 2.4:

where X is the mean and σ is the standard derivation. The factor of $1/\sqrt{2\pi}\sigma_x$ insures that the integral of the pdf over all values is unity as required. It is easy to prove that this is the case by completing the squares in the integration of the exponential (see problem 2.2).

The Gaussian distribution is unusual in that it is completely determined by its first two moments, X and σ. This is *not* typical of most turbulence distributions. Nonetheless, it is sometimes useful to approximate turbulence as being Gaussian, often because of the absence of simple alternatives.

It is straightforward to show by integrating by parts that all the even central moments above the second are given by the following recursive relationship,

$$< (x - X)^n >= (n-1)(n-3)...3.1\sigma^n \tag{2.24}$$

Thus the fourth central moment is $3\sigma^4$, the sixth is $15\sigma^6$, and so forth.

Exercise: Prove this.

The probability distribution corresponding to the Gaussian distribution can be obtained by integrating the Gaussian pdf from $-\infty$ to $x = c$; i.e.,

$$F_{xG}(c) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{c} e^{-(c'-X)^2/2\sigma^2} dc' \tag{2.25}$$

The integral is related to the erf-function tabulated in many standard tables.

2.3.4 Skewness and kurtosis

Because of their importance in characterizing the shape of the pdf, it is useful to define scaled versions of third and fourth central moments, the *skewness* and *kurtosis* respectively. The *skewness* is defined as third central moment divided by the three-halves power of the second; i.e.,

$$S = \frac{< (x - X)^3 >}{< (x - X)^2 >^{3/2}} \tag{2.26}$$

The *kurtosis* defined as the fourth central moment divided by the square of the second; i.e.,

$$K = \frac{< (x - X)^4 >}{< (x - X)^2 >^2} \tag{2.27}$$

Both these are easy to remember if you note the S and K must be dimensionless.

The pdf's in Figure 2.4 can be distinguished by means of their skewness and kurtosis. The random variable shown in (b) has a higher kurtosis than that in (a). Thus the kurtosis can be used as an indication of the tails of a pdf, a higher kurtosis indicating that relatively larger excursions from the mean are more probable. The skewnesses of (a) and (b) are zero, whereas those for (c) and (d) are non-zero. Thus, as its name implies, a non-zero skewness indicates a skewed or asymmetric pdf, which in turn means that larger excursions in one direction are more probable than in the other. For a Gaussian pdf, the skewness is zero and the kurtosis is equal to three (see problem 2.4). The flatness factor, defined as $(K - 3)$, is sometimes used to indicate deviations from Gaussian behavior.

Exercise: Prove that the kurtosis of a Gaussian distributed random variable is 3.

2.4 Multivariate Random Variables

2.4.1 Joint pdfs and joint moments

Often it is important to consider more than one random variable at a time. For example, in turbulence the three components of the velocity vector are interrelated and must be considered together. In addition to the *marginal* (or single variable) statistical moments already considered, it is necessary to consider the **joint** statistical moments.

For example if u and v are two random variables, there are three second-order moments which can be defined $< u^2 >$, $< v^2 >$, and $< uv >$. The product moment $< uv >$ is called the *cross-correlation* or *cross-covariance*. The moments $< u^2 >$ and $< v^2 >$ are referred to as the *covariances*, or just simply the *variances*. Sometimes $< uv >$ is also referred to as the *correlation*.

In a manner similar to that used to build-up the probability density function from its measurable counterpart, the histogram, a **joint probability density function** (or **jpdf**), B_{uv}, can be built-up from the *joint histogram*. Figure 2.5 illustrates several examples of jpdf's which have different cross-correlations. For convenience the fluctuating variables u' and v' can be defined as

$$u' = u - U \tag{2.28}$$
$$v' = v - V \tag{2.29}$$

where as before capital letters are used to represent the mean values. Clearly the fluctuating quantities u' and v' are random variables with zero mean.

A positive value of $< u'v' >$ indicates that u' and v' tend to vary together. A negative value indicates that when one variable is increasing the other tends to be decreasing. A zero value of $< u'v' >$ indicates that there is no correlation between u' and v'. As will be seen below, it does *not* mean that they are statistically independent.

It is sometimes more convenient to deal with values of the cross-variances which have been normalized by the appropriate variances. Thus the *correlation coefficient* is defined as:

$$\rho_{uv} \equiv \frac{< u'v' >}{[< u'^2 >< v'^2 >]^{1/2}} \tag{2.30}$$

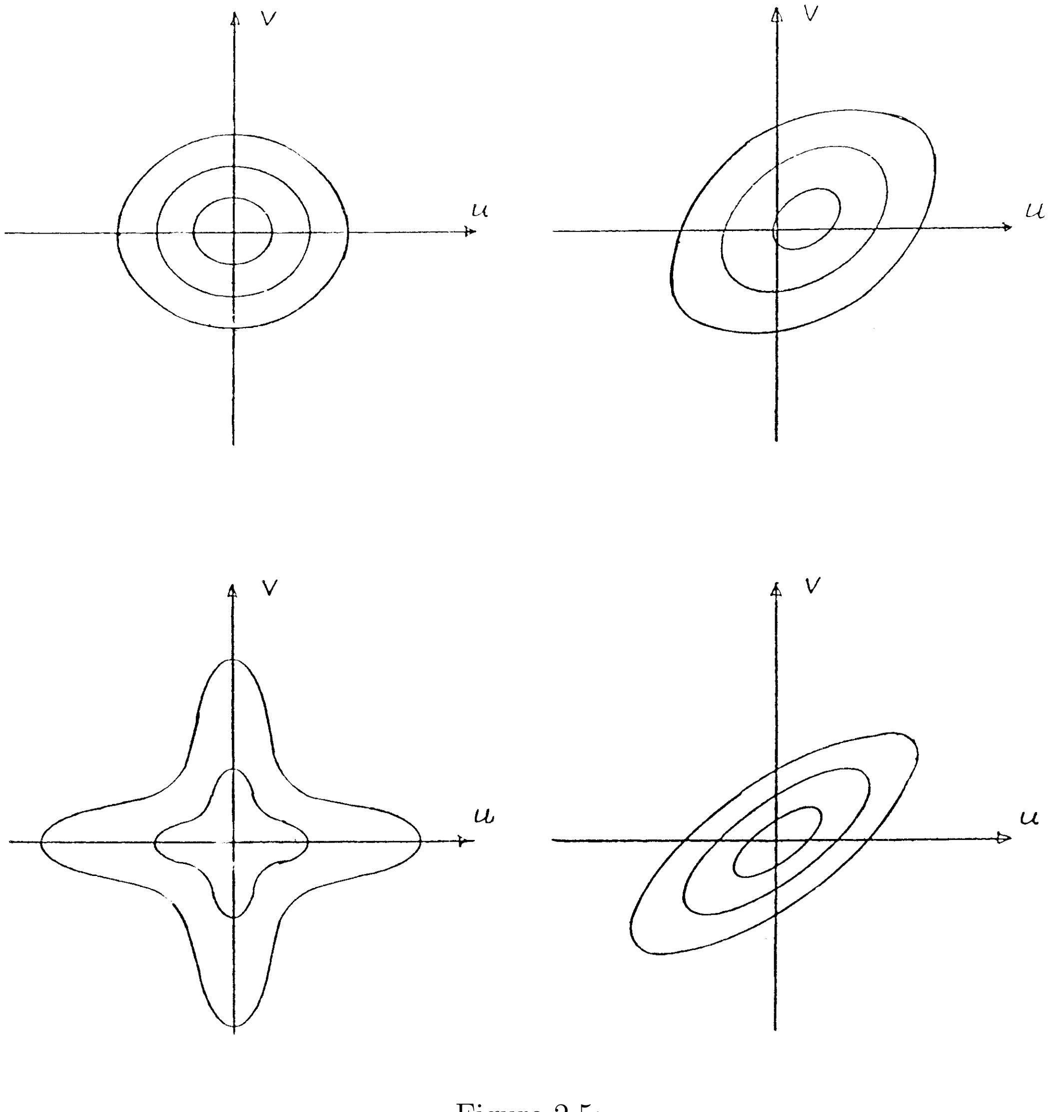

Figure 2.5:

The correlation coefficient is bounded by plus or minus one, the former representing perfect correlation and the latter perfect anti-correlation.

As with the single-variable pdf, there are certain conditions the joint probability density function must satisfy. If $B_{uv}(c_1, c_2)$ indicates the jpdf of the random variables u and v, then:

- Property 1:

$$B_{uv}(c_1, c_2) > 0 \tag{2.31}$$

  always.

- Property 2:

$$Prob\{c_1 < u < c_1 + dc_1, c_2 < v < c_2 + dc_2\} = B_{uv}(c_1, c_2)dc_1, dc_2 \tag{2.32}$$

- Property 3:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B_{uv}(c_1, c_2)dc_1 dc_2 = 1 \tag{2.33}$$

- Property 4:

$$\int_{-\infty}^{\infty} B_{uv}(c_1, c_2)dc_2 = B_u(c_1) \tag{2.34}$$

  where B_u is a function of c_1 only.

- Property 5:

$$\int_{-\infty}^{\infty} B_{uv}(c_1, c_2)dc_1 = B_v(c_2) \tag{2.35}$$

  where B_v is a function of c_2 only.

The functions B_u and B_v are called the *marginal probability density functions*, and they are simply the single variable pdf's defined earlier. The subscript is used to indicate which variable is left after the others are integrated out. Note that $B_u(c_1)$ is not the same as $B_{uv}(c_1, 0)$. The latter is only a slice through the c_2-axis, while the marginal distribution is weighted by the integral of the distribution of the other variable. Figure 2.6 illustrates these differences.

If the joint probability density function is known, the *joint moments* of all orders can be determined. Thus the m, n-th joint moment is

$$< u^m v^n >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_1^m c_2^n B_{uv}(c_1, c_2)dc_1 dc_2 \tag{2.36}$$

where m and n can take any value. The corresponding central-moment is:

$$< (u - U)^m (v - V)^n >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (c_1 - U)^m (c_2 - V)^n B_{uv}(c_1, c_2)dc_1 dc_2 \tag{2.37}$$

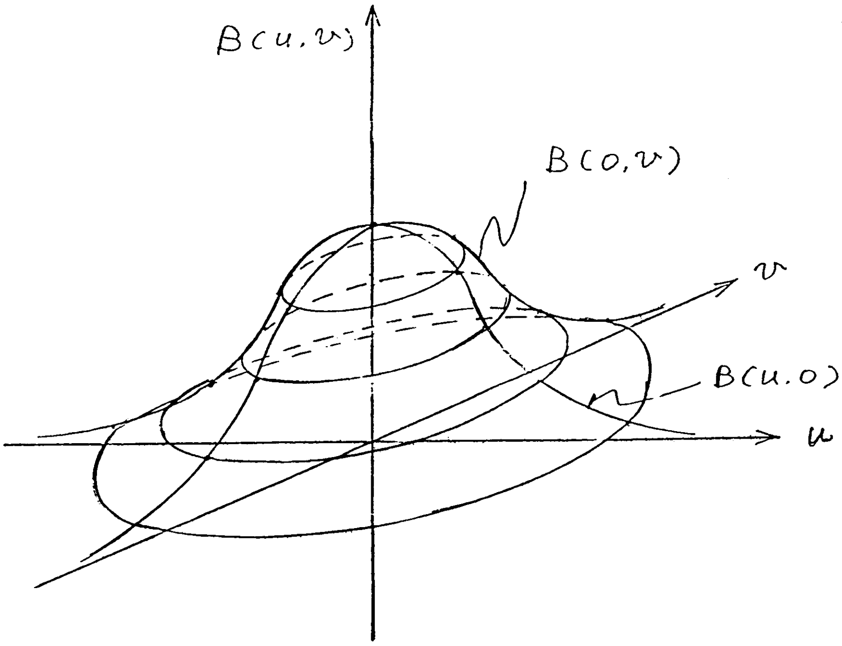

Figure 2.6:

In the preceding discussions, only two random variables have been considered. The definitions, however, can easily be generalized to accomodate any number of random variables. In addition, the joint statistics of a single random variable at different times or at different points in space could be considered. This will be done later when stationary and homogeneous random processes are considered.

2.4.2 The bi-variate normal (or Gaussian) distribution

If u and v are *normally* distributed random variables with standard deviations given by σ_u and σ_v, respectively, with correlation coefficient ρ_{uv}, then their joint probability density function is given by

$$B_{uvG}(c_1, c_2) = \frac{1}{2\pi\sigma_u\sigma_v} \exp\left[\frac{(c_1 - U)^2}{2\sigma_u^2} + \frac{(c_2 - V)^2}{2\sigma_v^2} - \rho_{uv}\frac{c_1 c_2}{\sigma_u \sigma_v}\right] \tag{2.38}$$

This distribution is plotted in Figure 2.7 for several values of ρ_{uv} where u and v are assumed to be identically distributed (i.e., $< u^2 >=< v^2 >$).

It is straightforward to show (by completing the square and integrating) that this yields the single variable Gaussian distribution for the marginal distributions (see problem 2.5). It is also possible to write a *multivariate Gaussian* probability density function for any number of random variables.

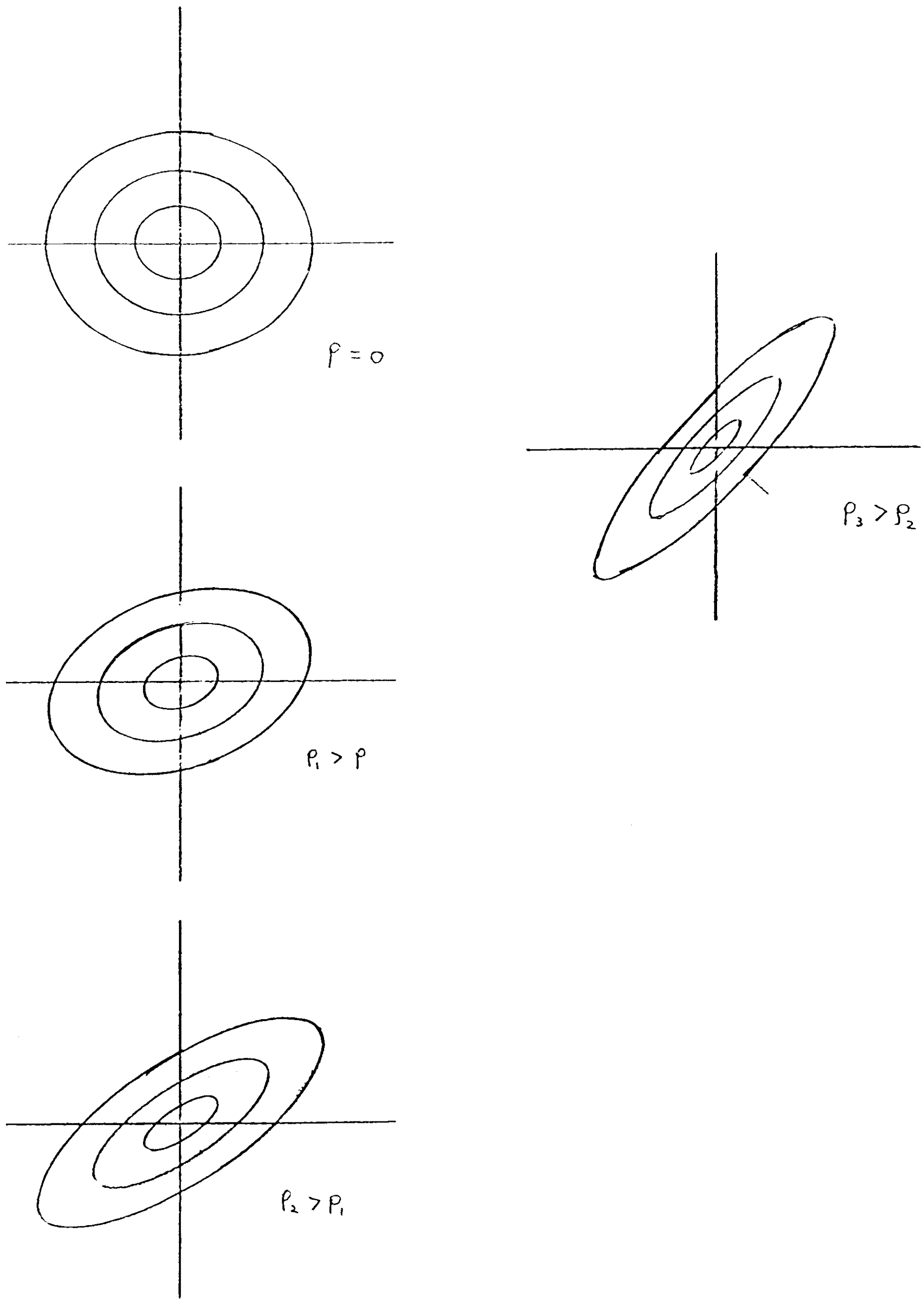

Figure 2.7:

Exercise: Prove that equation 2.23 results from integrating out the dependence of either variable using equations 2.34 or 2.35.

2.4.3 Statistical independence and lack of correlation

Definition: Statistical Independence Two random variables are said to be *statistically independent* if their joint probability density is equal to the product of their marginal probability density functions. That is,

$$B_{uv}(c_1, c_2) = B_u(c_1)B_v(c_2) \tag{2.39}$$

It is easy to see that statistical independence implies a complete lack of correlation; i.e., $\rho_{uv} \equiv 0$. From the definition of the cross-correlation,

$$\begin{aligned}
< (u-U)(v-V) > &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(c_1-U)(c_2-V)B_{uv}(c_1,c_2)dc_1dc_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(c_1-U)(c_2-V)B_u(c_1)B_v(c_2)dc_1dc_2 \\
&= \int_{-\infty}^{\infty}(c_1-U)B_u(c_1)dc_1\int_{-\infty}^{\infty}(c_2-V)B_v(c_2)dc_2 \\
&= 0
\end{aligned} \tag{2.40}$$

where we have used equation 2.39 since the first central moments are zero by definition.

It is important to note that the inverse is not true — *lack of correlation does not imply statistical independence!* To see this consider two identically distributed random variables, u' and v', which have zero means and a non-zero correlation $< u'v' >$. From these two correlated random variables two other random variables, x and y, can be formed as:

$$x = u' + v' \tag{2.41}$$

$$y = u' - v' \tag{2.42}$$

Clearly x and y are *not* statistically independent since the quantities from which they were formed are not statistically independent. They are, however, *uncorrelated* because:

$$\begin{aligned}
< xy > &= < (u'+v')(u'-v') > \\
&= < u'^2 > + < u'v' > - < u'v' > - < v'^2 > \\
&= 0
\end{aligned} \tag{2.43}$$

since u' and v' are identically distributed (and as a consequence $< u'^2 > = < v'^2 >$).

Figure 2.8 illustrates the change of variables carried out above. The jpdf resulting from the transformation is symmetric about both axes, thereby eliminating the correlation. Transformation, however, does not insure that the distribution is separable, i.e., $B_{x,y}(a_1, a_2) = B_x(a_1)B_y(a_2)$, as required for statistical independence.

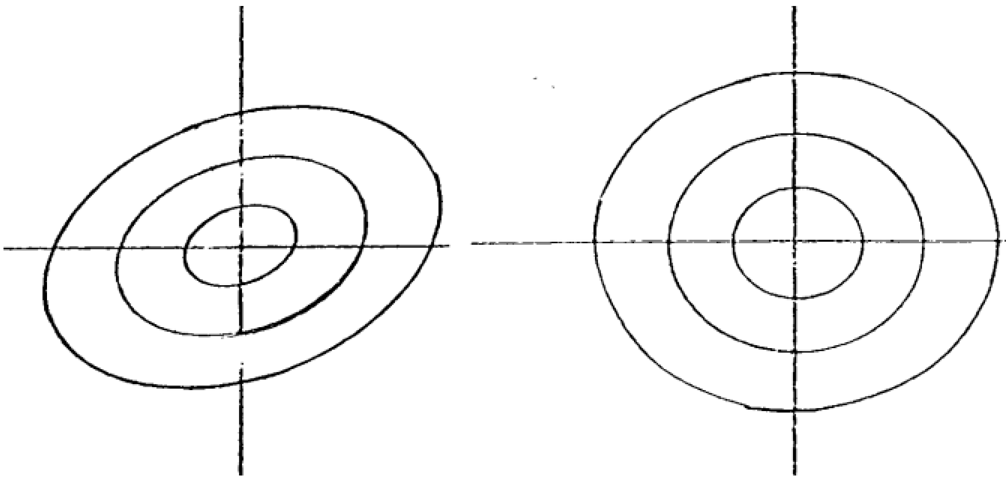

Figure 2.8:

2.5 Estimation from a Finite Number of Realizations

2.5.1 Estimators for averaged quantities

Since there can never an infinite number of realizations from which ensemble averages (and probability densities) can be computed, it is essential to ask: *How many realizations are enough?* The answer to this question must be sought by looking at the statistical properties of estimators based on a finite number of realizations. There are two questions which must be answered. The first one is:

- Is the expected value (or mean value) of the estimator equal to the true ensemble mean? Or in other words, is the estimator *unbiased?*

The second question is:

- Does the difference between the value of the estimator and that of the true mean decrease as the number of realizations increases? Or in other words, does the estimator *converge* in a statistical sense (or converge in probability). Figure 2.9 illustrates the problems which can arise.

2.5.2 Bias and convergence of estimators

A procedure for answering these questions will be illustrated by considering a simple **estimator** for the mean, the arithmetic mean considered above, X_N. For N independent realizations, x_n, $n = 1, 2, \cdots, N$ where N is finite, X_N is given by:

$$X_N = \frac{1}{N} \sum_{n=1}^{N} x_n \tag{2.44}$$

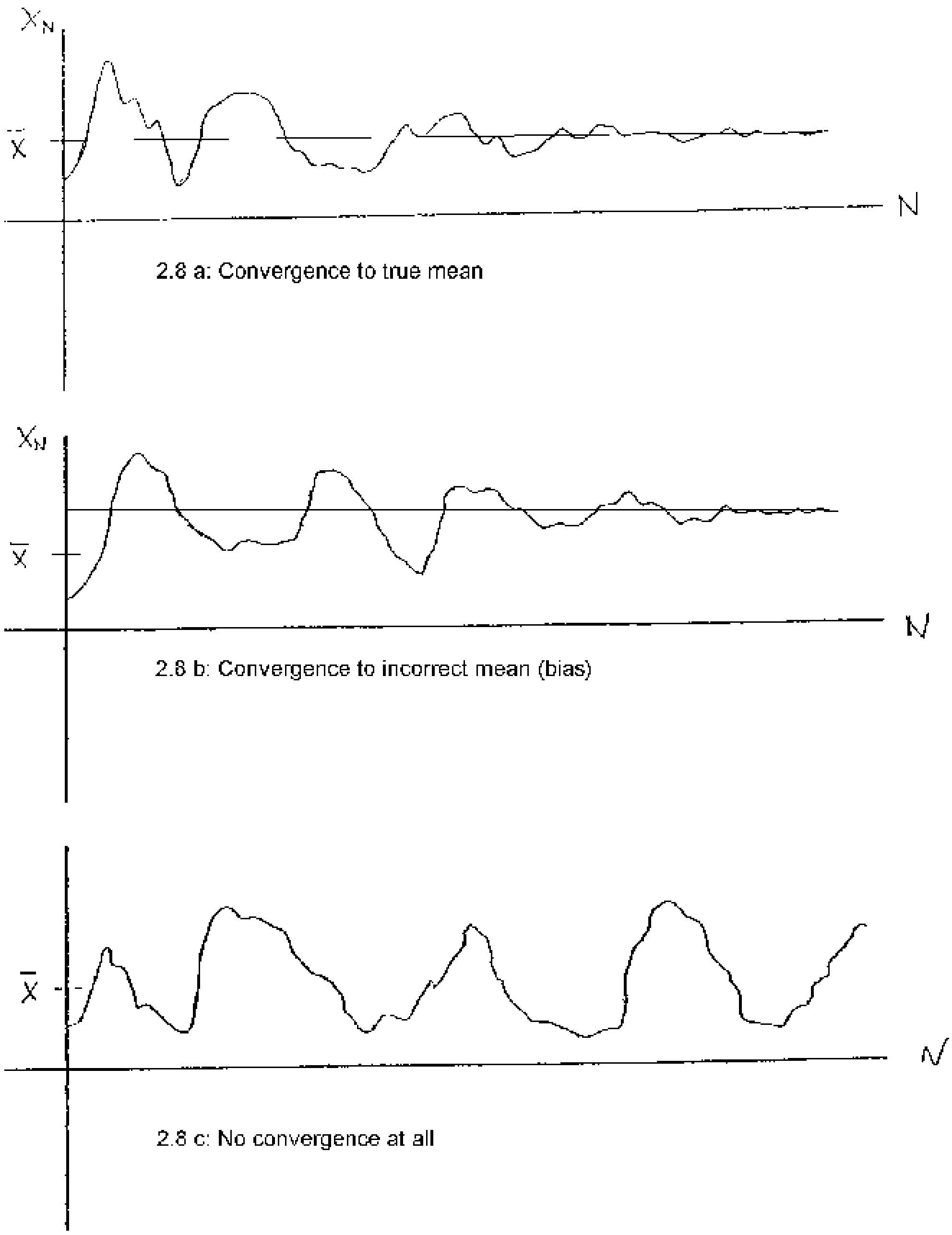
X_N
$\bar{X}$
N
2.8 a: Convergence to true mean
X_N
$\bar{X}$
N
2.8 b: Convergence to incorrect mean (bias)
$\bar{X}$
N
2.8 c: No convergence at all

Figure 2.9:

Now, as we observed in our simple coin-flipping experiment, since the x_n are random, so must be the value of the estimator X_N. For the estimator to be *unbiased*, the mean value of X_N must be the true ensemble mean, X; i.e.,

$$lim_{N\to\infty} X_N = X \tag{2.45}$$

It is easy to see that since the operations of averaging and adding commute,

$$\begin{aligned} < X_N > &= \left\langle \frac{1}{N} \sum_{n=1}^{N} x_n \right\rangle & (2.46) \\ &= \frac{1}{N} \sum_{n=1}^{N} < x_n > & (2.47) \\ &= \frac{1}{N} N X = X & (2.48) \end{aligned}$$

(Note that the expected value of each x_n is just X since the x_n are assumed identically distributed). Thus x_N is, in fact, an *unbiased estimator for the mean.*

The question of *convergence* of the estimator can be addressed by defining the square of **variability of the estimator**, say $\epsilon^2_{X_N}$, to be:

$$\epsilon^2_{X_N} \equiv \frac{var\{X_N\}}{X^2} = \frac{< (X_N - X)^2 >}{X^2} \tag{2.49}$$

Now we want to examine what happens to ϵ_{X_N} as the number of realizations increases. For the estimator to converge it is clear that ϵ_x should decrease as the number of samples increases. Obviously, we need to examine the variance of X_N first. It is given by:

$$\begin{aligned} var\{X_N\} &= < X_N - X^2 > \\ &= \left\langle \left[\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (x_n - X) \right]^2 \right\rangle - X^2 & (2.50) \end{aligned}$$

since $< X_N >= X$ from equation 2.46. Using the fact that operations of averaging and summation commute, the squared summation can be expanded as follows:

$$\begin{aligned} \left\langle \left[\lim_{N\to\infty} \sum_{n=1}^{N} (x_n - X) \right]^2 \right\rangle &= \lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} < (x_n - X)(x_m - X) > \\ &= \lim_{N\to\infty} \frac{1}{N^2} \sum_{n=1}^{N} < (x_n - X)^2 > \\ &= \frac{1}{N} var\{x\}, & (2.51) \end{aligned}$$

where the next to last step follows from the fact that the x_n are assumed to be statistically independent samples (and hence uncorrelated), and the last step from the definition of the variance. It follows immediately by substitution into equation 2.49 that the square of the variability of the estimator, X_N, is given by:

$$\begin{aligned}\epsilon^2_{X_N} &= \frac{1}{N}\frac{var\{x\}}{X^2} \\ &= \frac{1}{N}\left[\frac{\sigma_x}{X}\right]^2 \end{aligned} \tag{2.52}$$

Thus *the variability of the estimator depends inversely on the number of independent realizations, N, and linearly on the relative fluctuation level of the random variable itself, σ_x/X.* Obviously if the relative fluctuation level is zero (either because there the quantity being measured is constant and there are no measurement errors), then a single measurement will suffice. On the other hand, as soon as there is any fluctuation in the x itself, the greater the fluctuation (relative to the mean of x, $< x >= X$), then the more independent samples it will take to achieve a specified accuracy.

Example: In a given ensemble the relative fluctuation level is 12% (i.e., $\sigma_x/X = 0.12$). What is the fewest number of independent samples that must be acquired to measure the mean value to within 1%?

Answer Using equation 2.52, and taking $\epsilon_{X_N} = 0.01$, it follows that:

$$(0.01)^2 = \frac{1}{N}(0.12)^2 \tag{2.53}$$

or $N \geq 144$.

2.6 Generalization to the estimator of any quantity

Similar relations can be formed for the estimator of any function of the random variable, say $f(x)$. For example, an estimator for the average of f based on N realizations is given by:

$$F_N \equiv \frac{1}{N}\sum_{n=1}^{N} f_n \tag{2.54}$$

where $f_n \equiv f(x_n)$. It is straightforward to show that this estimator is unbiased, and its variability (squared) is given by:

$$\epsilon^2_{F_N} = \frac{1}{N}\frac{var\{f(x)\}}{< f(x) >^2} \tag{2.55}$$

Example: Suppose it is desired to estimate the variability of an estimator for the variance based on a finite number of samples as:

$$var_N\{x\} \equiv \frac{1}{N}\sum_{n=1}^{N}(x_n - X)^2 \tag{2.56}$$

(Note that this estimator is not really very useful since it presumes that the mean value, X, is known, whereas in fact usually only X_N is obtainable as in Problem 2.6 below.)

Answer Let $f = (x - X)^2$ in equation 2.55 so that $F_N = var_N\{x\}$, $< f >= var\{x\}$ and $var\{f\} = var\{(x - X)^2 - var[x - X])\}$. Then:

$$\epsilon^2_{var_N} = \frac{1}{N} \frac{var\{(x - X)^2 - var[x]\}}{(var\{x\})^2} \tag{2.57}$$

This is easiest to understand if we first expand only the numerator to obtain:

$$var\{(x - X)^2 - var[x]\} =< (x - X)^4 > -[var\{x\}]^2 \tag{2.58}$$

Thus

$$\epsilon^2_{var_N} = \frac{< (x - X)^4 >}{[var\{x\}]^2} - 1 \tag{2.59}$$

Obviously to proceed further we need to know how the fourth central moment relates to the second central moment. As noted earlier, in general this is *not* known. If, however, it is reasonable to assume that x is a Gaussian distributed random variable, we know from section 2.3.4 that the kurtosis is 3. Then for Gaussian distributed random variables,

$$\epsilon^2_{var_N} = \frac{2}{N} \tag{2.60}$$

Thus the number of independent data required to produce the same level of convergence for an estimate of the variance of a Gaussian distributed random variable is $\sqrt{2}$ times that of the mean. It is easy to show that the higher the moment, the more the amount of data required (see Problem 2.7).

As noted earlier, turbulence problems are not usually Gaussian, and in fact values of the kurtosis substantially greater than 3 are commonly encountered, especially for the moments of differentiated quantities. Clearly the non-Gaussian nature of random variables can affect the planning of experiments, since substantially greater amounts of data can be required to achieved the necessary statistical accuracy.

Problems for Chapter 2

2.1 By using the definition of the probability density function as the limit of the histogram of a random variable as the internal size goes to zero and as the number of realizations becomes infinite (equation 2.12), show that the probability average defined by equation 2.17 and the ensemble average defined by equation 2.2 are the same.

2.2 By completing the square in the exponential, prove that the pdf for the normal distribution given by equation 2.23 integrates to unity (equation 2.16)

2.3 Prove equation 2.24.

2.4 Prove for a normal distribution that the skewness is equal to zero and that the kurtosis is equal to three.

2.5 Show by integrating over one of the variables that the Gaussian jpdf given by equation 2.38 integrates to the marginal distribution pdf given by equation 2.23, regardless of the value of the correlation coefficient.

2.6 Find the variability of an estimator for the variance using equation 2.58, but with the sample mean, X_N, substituted for the true mean, X.

2.7 Create a simple estimator for the fourth central moment — assuming the second to be known exactly. Then find its variability for a Gaussian distributed random variable.

2.8 You are attempting to measure a Gaussian distributed random variable with 12 bit A/D converter which can only accept voltage inputs between 0 and 10. Assume the mean voltage is +4, and the rms voltage is 4. Show what a histogram of your measured signal would look like assuming any voltage which is clipped goes into the first or last bins. Also compute the first three moments (central) of the measured signal.

Chapter 3

The Reynolds Averaged Equations and the Turbulence Closure Problem

3.1 The Equations Governing the Instantaneous Fluid Motions

All fluid motions, whether turbulent or not, are governed by the dynamical equations for a fluid. These can be written using Cartesian tensor notation as:

$$\rho \left[\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} \right] = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial \tilde{T}_{ij}^{(v)}}{\partial x_j} \tag{3.1}$$

$$\left\{ \frac{\partial \tilde{\rho}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\rho}}{\partial x_j} \right\} + \tilde{\rho} \frac{\partial \tilde{u}_j}{\partial x_j} = 0 \tag{3.2}$$

where $\tilde{u}_i(\vec{x}, t)$ represents the i-th component of the fluid velocity at a point in space, $[\vec{x}]_i = x_i$, and time, t. Also $\tilde{p}(\vec{x}, t)$ represents the static pressure, $\tilde{T}_{ij}^{(v)}(\vec{x}, t)$, the viscous (or deviatoric) stresses, and $\tilde{\rho}$ the fluid density. The tilde over the symbol indicates that an instantaneous quantity is being considered. Also the Einstein summation convention has been employed.[1]

In equation 3.1, the subscript i is a free index which can take on the values 1, 2, and 3. Thus equation 3.1 is in reality three separate equations. These three equations are just Newton's second law written for a continuum in a spatial (or Eulerian) reference frame. Together they relate the rate of change of momentum per unit mass (ρu_i), a vector quantity, to the contact and body forces.

Equation 3.2 is the equation for mass conservation in the absence of sources (or sinks) of mass. Almost all flows considered in this book will be incompressible, which implies that the derivative of the density following the fluid material [the

[1] Einstein summation convention: repeated indices in a single term are summed over 1,2, and 3.

term in brackets] is zero. Thus for incompressible flows, the mass conservation equation reduces to:

$$\frac{D\tilde{\rho}}{Dt} = \frac{\partial \tilde{\rho}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\rho}}{\partial x_j} = 0 \tag{3.3}$$

From equation 3.2 it follows that for incompressible flows,

$$\frac{\partial \tilde{u}_i}{\partial x_j} = 0 \tag{3.4}$$

The viscous stresses (the stress minus the mean normal stress) are represented by the tensor $\tilde{T}_{ij}^{(v)}$. From its definition, $\tilde{T}_{kk}^{(v)} = 0$. In many flows of interest, the fluid behaves as a Newtonian fluid in which the viscous stress can be related to the fluid motion by a constitutive relation of the form

$$\tilde{T}_{ij}^{(v)} = 2\mu \left[\tilde{s}_{ij} - \frac{1}{3} \tilde{s}_{kk} \delta_{ij} \right] \tag{3.5}$$

The viscosity, μ, is a *property of the fluid* that can be measured in an independent experiment. $\tilde{s}_{ij}$ is the instantaneous strain rate tensor defined by

$$\tilde{s}_{ij} \equiv \frac{1}{2} \left[\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right] \tag{3.6}$$

From its definition, $\tilde{s}_{kk} = \partial \tilde{u}_k / \partial x_k$. If the flow is incompressible, $\tilde{s}_{kk} = 0$ and the Newtonian constitutive equation reduces to

$$\tilde{T}_{ij}^{(v)} = 2\mu \tilde{s}_{ij} \tag{3.7}$$

Throughout this text, *unless explicity stated otherwise*, the density, $\tilde{\rho} = \rho$ and the viscosity μ will be assumed constant. With these assumptions, the instantaneous momentum equations for a Newtonian fluid reduce to:

$$\left[\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} \right] = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} \tag{3.8}$$

where the kinematic viscosity, ν, has been defined as:

$$\nu \equiv \frac{\mu}{\rho} \tag{3.9}$$

Note that since the density is assumed constant, the tilde is no longer necessary.

Sometimes it will be more instructive and convenient to *not* explicitly include incompressibility in the stress term, but to refer to the incompressible momentum equation in the following form:

$$\rho \left[\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} \right] = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial \tilde{T}_{ij}^{(v)}}{\partial x_j} \tag{3.10}$$

This form has the advantage that it is easier to keep track of the exact role of the viscous stresses.

3.2 Equations for the Average Velocity

Turbulence is that chaotic state of motion characteristic of solutions to the equations of motion at high Reynolds number. Although laminar solutions to the equations often exist that are consistent with the boundary conditions, perturbations to these solutions (sometimes even infinitesimal) can cause them to become turbulent. To see how this can happen, it is convenient to analyze the flow in two parts, a mean (or average) component and a fluctuating component. Thus the instantaneous velocity and stresses can be written as:

$$\begin{aligned} \tilde{u}_i &= U_i + u_i \\ \tilde{p} &= P + p \\ \tilde{T}_{ij}^{(v)} &= T_{ij}^{(v)} + \tau_{ij}^{(v)} \end{aligned} \tag{3.11}$$

where U_i, p, and $T_{ij}^{(v)}$ represent the mean motion, and u_i, p, and τ_{ij} the fluctuating motions. This technique for decomposing the instantaneous motion is referred to as the *Reynolds decomposition*. Note that if the averages are defined as ensemble means, they are, in general, *time-dependent*. For the remainder of this book, unless otherwise stated, the density will be assumed constant so $\tilde{\rho} \equiv \rho$ and its fluctuation is zero.

Substitution of equations 3.11 into equations 3.10 yields

$$\rho \left[\frac{\partial (U_i + u_i)}{\partial t} + (U_j + u_j) \frac{\partial (U_i + u_i)}{\partial x_j} \right] = - \frac{\partial (P + p)}{\partial x_i} + \frac{\partial (T_{ij}^{(v)} + \tau_{ij}^{(v)})}{\partial x_j} \tag{3.12}$$

This equation can now be averaged to yield an equation expressing momentum conservation for the averaged motion. Note that the operations of averaging and differentiation commute; i.e., the average of a derivative is the same as the derivative of the average. Also, the average of a fluctuating quantity is zero.[2] Thus the equation for the averaged motion reduces to:

$$\rho \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] = - \frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}^{(v)}}{\partial x_j} - \rho \left\langle u_j \frac{\partial u_i}{\partial x_j} \right\rangle \tag{3.13}$$

where the remaining fluctuating product term has been moved to the right-hand side of the equation. Whether or not this last term is zero like the other fluctuating terms depends on the correlation of terms in the product. In general, these correlations are *not* zero.

The mass conservation equation can be similarly decomposed. In incompressible form, substitution of equations 3.11 into equation 3.4 yields:

$$\frac{\partial (U_j + u_j)}{\partial x_j} = 0 \tag{3.14}$$

[2] These are easily proven from the definitions of both.

of which the average is:

$$\frac{\partial U_j}{\partial x_j} = 0 \tag{3.15}$$

It is clear from equation 3.15 that the averaged motion satisfies the same form of the mass conservation equation as does the instantaneous motion, at least for incompressible flows. How much simpler the turbulence problem would be if the same were true for the momentum! Unfortunately, as is easily seen from equation 3.13, such is not the case.

Equation 3.15 can be subtracted from equation 3.14 to yield an equation for the instantaneous motion alone; i.e.,

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{3.16}$$

Again, like the mean, the form of the original instantaneous equation is seen to be preserved. The reason, of course, is obvious: the continuity equation is linear. The momentum equation, on the other hand, is not; hence the difference.

Equation 3.16 can be used to rewrite the last term in equation 3.13 for the mean momentum. Multiplying equation 3.16 by u_i and averaging yields:

$$\left\langle u_i \frac{\partial u_j}{\partial x_j} \right\rangle = 0 \tag{3.17}$$

This can be added to $< u_j \partial u_i / \partial x_j >$ to obtain:

$$\left\langle u_j \frac{\partial u_i}{\partial x_j} \right\rangle + 0 = \left\langle u_j \frac{\partial u_i}{\partial x_j} \right\rangle + \left\langle u_i \frac{\partial u_j}{\partial x_j} \right\rangle = \frac{\partial}{\partial x_j} < u_i u_j > \tag{3.18}$$

where again the fact that arithmetic and averaging operations commute has been used.

The equation for the averaged momentum, equation 3.13 can now be rewritten as:

$$\rho \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] = -\frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}^{(v)}}{\partial x_j} - \frac{\partial}{\partial x_j} (\rho < u_i u_j >) \tag{3.19}$$

The last two terms on the right-hand side are both divergence terms and can be combined; the result is:

$$\rho \left[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right] = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[T_{ij}^{(v)} - \rho < u_i u_j > \right] \tag{3.20}$$

Now the terms in square brackets on the right have the dimensions of stress. The first term is, in fact, the *viscous stress.* The second term, on the other hand, is not a stress at all, but simply a re-worked version of the fluctuating contribution to the non-linear acceleration terms. The fact that it can be written this way, however, indicates that *at least as far as the mean motion is concerned*, it *acts* as though it were a *stress* — hence its name, the **Reynolds stress**. In the succeeding sections the consequences of this difference will be examined.

3.3 The Turbulence Problem

It is the appearance of the Reynolds stress which makes the turbulence problem so difficult — at least from the engineers perspective. Even though we can pretend it is a stress, the physics which give rise to it are very different from the viscous stress. The viscous stress can be related directly to the other flow properties by constitutive equations, which in turn depend only on the properties of the *fluid* (as in equation 3.5 for a Newtonian fluid). The reason this works is that when we make such closure approximations for a fluid, we are averaging over characteristic length and time scales much smaller than those of the *flows* we are interested in. Yet at the same time, these scales are much larger than the *molecular* length and time scales which characterize the molecular interactions that are actually causing the momentum transfer. (This is what the continuum approximation is all about.)

The *Reynolds stress*, on the other hand, arises directly from the *flow* itself! *Worse, the scales of the fluctuating motion which give rise to it* **are** *the scales we are interested in.* This means that the closure ideas which worked so well for the viscous stress, should not be expected to work too well for the Reynolds stress. And as we shall see, they do not.

This leaves us in a terrible position. Physics and engineering are all about writing equations (and boundary conditions) so we can solve them to make predictions. We don't want to have to build prototype airplanes first to see if they will fall out of the sky. Instead we want to be able to analyze our designs *before* building, to save the cost in money and lives if our ideas are wrong. The same is true for dams and bridges and tunnels and automobiles. If we had confidence in our turbulence models, we could even build huge one-offs and expect them to work the first time. Unfortunately, even though turbulence models have improved to the point where we can use them in design, we still cannot trust them enough to eliminate expensive wind tunnel and model studies. And recent history is full of examples to prove this.

The turbulence problem (from the engineers perspective) is then three-fold:

- **The averaged equations are not closed.** Count the unknowns in equation 3.20 above. Then count the number of equations. Even with the continuity equation we have at least six equations too few.

- **The simple ideas to provide the extra equations usually do not work.** And even when we can fix them up for a particular class of flows (like the flow in a pipe, for example), they will most likely not be able to predict what happens in even a slightly different environment (like a bend).

- **Even the last resort of compiling engineering tables for design handbooks carries substantial risk.** This is the last resort for the engineer who lacks equations or cannot trust them. Even when based on a wealth of experience, they require expensive model testing to see if they can be extrapolated to a particular situation. Often they cannot, so infinitely clever is Mother Nature in creating turbulence that is unique to a particular set of boundary conditions.

Turbulent flows are indeed flows!. And that is the problem.

3.4 The Origins of Turbulence

Turbulent flows can often be observed to arise from laminar flows as the Reynolds number, (or some other relevant parameter) is increased. This happens because small distubances to the flow are no longer damped by the flow, but begin to grow by taking energy from the original laminar flow. This natural process is easily visualized by watching the simple stream of water from a faucet (or even a pitcher). Turn the flow on very slowly (or pour) so the stream is very smooth initially, at least near the outlet. Now slowly open the faucet (or pour faster) and observe what happens, first far away, then closer to the spout. The surface begins to exhibit waves or ripples which appear to grow downstream. In fact, they are growing by extracting energy from the primary flow. Eventually they grow enough that the flow breaks into drops. These are capillary instabilities arising from surface tension, but regardless of the type of instability, the idea is the same — small (or even infinitesimal) disturbances have grown to disrupt the serenity (and simplicity) of laminar flow.

The manner in which instabilities grow naturally in a flow can be examined using the equations we have already developed above. We derived them by decomposing the motion into a mean and a fluctuating part. But suppose instead we had decomposed the motion into a *base* flow part (the initially laminar part) and into a *disturbance* which represents a fluctuating part superimposed on the base flow. The result of substituting such a decomposition into the full Navier-Stokes equations and averaging is precisely that given by equations 3.13 and 3.15. But the very important difference is the additional restriction that what was previously identified as *the mean (or averaged) motion is now also the base or laminar flow.*

Now if the base flow is really a laminar flow (which it must be by our original hypothesis), then our averaged equations governing the base flow must yield the same mean flow solution as the original laminar flow on which the disturbance was superimposed. But this can happen only if these new averaged equations reduce to **exactly** the same laminar flow equations without any evidence of a disturbance. Clearly from equations 3.13 and 3.15, this can happen *only if all the Reynolds stress terms vanish identically*! Obviously this requires that the disturbances be infinitesimal so the extra terms can be neglected — hence our interest in infinitesimal disturbances.

So we hypothesized a base flow which was laminar and showed that it is unchanged even with the imposition of infinitesimal disturbances on it — *but only as long as the disturbances* **remain** *infinitesimal!* What happens if the disturbance starts to grow? Obviously before we conclude that all laminar flows are laminar forever we better investigate whether or not these infinitesimal disturbances can grow to *finite* size. To do this we need an equation for the fluctuation itself.

An equation for the fluctuation (which might be an imposed disturbance) can be obtained by subtracting the equation for the mean (or base) flow from that for the instantaneous motion. We already did this for the continuity equation. Now we will do it for the momentum equation. Subtracting equation 3.13 from equation 3.11 yields an equation for the fluctuation as:

$$\rho \left[\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} \right] = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}^{(v)}}{\partial x_j} - \rho \left[u_j \frac{\partial U_i}{\partial x_j} \right] - \left\{ u_j \frac{\partial u_i}{\partial x_j} - \rho \left\langle u_j \frac{\partial u_i}{\partial x_j} \right\rangle \right\} \quad (3.21)$$

It is very important to note the type and character of the terms in this equation. First note that the left-hand side is the derivative of the *fluctuating* velocity following the *mean* motion. This is exactly like the term which appears on the left-hand side of the equation for the mean velocity, equation 3.13. The first two terms on the right-hand side are also like those in the mean motion, and represent the fluctuating pressure gradient and the fluctuating viscous stresses. The third term on the right-hand side is new, and will be seen later to represent the primary means by which fluctuations (and turbulence as well!) extract energy from the mean flow, the so-called *production terms*. The last term is quadratic in the fluctuating velocity, unlike all the others which are linear. Note that all of the terms vanish identically if the equation is averaged, the last because its mean is subtracted from it.

Now we want to examine what happens if the disturbance is small. In the limit as the amplitude of the disturbance (or fluctuation) is *infinitesimal*, the bracketed term in the equation for the fluctuation vanishes (since it involves products of infinitesimals), and the remaining equation is *linear in the disturbance*. The study of whether or not such infinitesimal disturbances can grow is called **Linear Fluid Dynamic Stability Theory**. These linearized equations are very different from those governing turbulence. Unlike the equations for disturbances of *finite* amplitude, the linearized equations are well-posed (or closed) since the Reynolds stress terms are gone.

The absence of the non-linear terms, however, constrains the validity of the linear analysis to only the initial stage of disturbance growth. This is because as soon as the fluctuations begin to grow, their amplitudes can no longer be assumed infinitesimal and the Reynolds stress (or more properly, the non-linear fluctuating terms), become important. As a result the base flow equations begin to be modified so that the solution to them can no longer be identical to the laminar flow (or base flow) from which it arose. Thus while linear stability theory can predict *when* many flows become *unstable*, it can say very little about *transition to turbulence* since this process is highly non-linear.

It is also clear from the above why the process of transition to turbulence is so dependent on the state of the background flow. If the disturbances present in the base flow are small enough, then Linear Stability Theory will govern their evolution. On the other hand if the disturbances to the base flow are not small enough, Linear Stability Theory can never apply since the non-linear terms will never be negligible. This is so-called *by-pass transition*. It is not uncommon to encounter situations like this in engineering environments where the incoming flow has a modest turbulence level super-imposed upon it. In such cases, the nature of the disturbances present is as important as their intensities, with the consequence that a general transition criterion may not exist, and perhaps should not even be expected.

3.5 The importance of non-linearity

We saw in the preceding section that non-linearity was one of the essential features of turbulence. When small disturbances grow large enough to interact *with each other*, we enter a whole new world of complex behavior. Most of the rules we learned for linear systems do not apply. Since most of your mathematical training has been for linear equations, most of your mathematical intuition therefore will not apply either. On the other hand, you may surprise yourself by discovering how much your *non-mathematical* intuition already recognizes non-linear behavior and accounts for it.

Consider the following simple example. Take a long stick with one person holding each end and stand at the corner of a building. Now place the middle of the stick against the building and let each person apply pressure in the same direction so as to bend the stick. If the applied force is small, the stick deflects (or bends) a small amount. Double the force, and the deflection is approximately doubled. Quadruple the force and the deflection is quadrupled. Now you don't need a Ph.D. in Engineering to know what is going to happen if you continue this process. **The stick is going to break!**

But where in the equations for the deflection of the stick is there anything that predicts this can happen? Now if you are thinking only like an engineer, you are probably thinking: he's asking a stupid question. Of course you can't continue to increase the force because you will exceed first the yield stress, then the breaking limit, and of course the stick will break.

But pretend I am the company president with nothing more than an MBA.[3] I don't know much about these things, but you have told me in the past that your computers have equations to predict everything. So I repeat: Where in the equations for the deflection of this stick does it tell me this is going to happen?

The answer is very simple: **There is *nothing* in the equations that will predict this.** And the reason is also quite simple: You lost the ability to predict catastrophes like breaking when you linearized the fundamental equations —

[3]For some reason the famous o-ring disaster of the the Challenger space shuttle comes to mind here.

which started out as Newton's Law too. In fact, before linearization, they were exactly the same as those for a fluid, only the constitutive equation was different.

If we had NOT linearized these equations and had constitutive equations that were more general, then we possibly could apply these equation right to and past the limit. The point of fracture would be a bifurcation point for the solution.

Now the good news is that for things like reasonable deflections of beams, linearization works wonderfully since we hope most things we build don't deflect too much — especially if you are sitting on a fault as I am at this moment.[4] Unfortunately, as we noted above, for fluids the disturbances tend to quickly become dominated by the non-linear terms. This, of course, means our linear analytical techniques are pretty useless for fluid mechanics, and especially turbulence.

But all is not lost. Just as we have learned to train ourselves to anticipate when sticks break, we have to train ourselves to anticipate how non-linear fluid phenomena behave. Toward that end we will consider two simple examples: one from algebra — the logistic map, and one from fluid mechanics — simple vortex stretching.

Example 1: An experiment with the logistic map.

Consider the behavior of the simple equation:

$$y_{n+1} = ry_n(1 - y_n) \tag{3.22}$$

where $n = 1, 2, \cdots$, $0 < y < 1$ and $r > 0$. The idea is that you pick any value for y_1, use the equation to find y_2, then insert that value on the right-hand side to find y_3, and just continue the process as long as you like. Make sure you note any dependence of the final result on the initial value for y.

- First notice what happens if you linearize this equation by disregarding the term in parentheses; i.e., consider the simpler equation $y_{n+1} = ry_n$. My guess is that you won't find this too exciting — unless, of course, you are one of those rare individuals who likes watching grass grow.

- Now consider the full equation and note what happens for $r < 3$, and especially what happens for very small values of r. Run as many iterations as necessary to make sure your answer has converged. Do NOT try to take short-cuts by programming all the steps at once. Do them one at a time so you can see what is happening. Believe me, it will be much easier this way in the long run.

- Now research carefully what happens when $r = 3.1, 3.5$, and 3.8. Can you recognize any patterns.

[4] I am sitting at the moment of this writing at the Institute for Theoretical Physics at the University of California/Santa Barbara.

- Vary r between 3 and 4 to see if you can find the boundaries for what you are observing.
- Now try values of $r > 4$. How do you explain this?

Example 2: Stretching of a simple vortex.
Imagine a simple vortex filament that looks about like a strand of spaghetti. Now suppose it is in an otherwise steady inviscid incompressible flow. Use the vorticity equation to examine the following:

- Examine first what happens to it in two-dimensional velocity field. Note particularly whether any new vorticity can be produced; i.e., can the material derivative of the vorticity ever be greater than zero? (Hint: look at the $\omega_j \partial u_i / \partial x_j$-term.)
- Now consider the same vortex filament in a three-dimensional flow. Note particularly the various ways new vorticity can be produced — if you have some to start with! Does all this have anything to do with non-linearities?

Now you are ready for a real flow.

A Simple Experiment: The Starbucks[5] problem

Go to the nearest coffee pot (or your favorite coffee shop) and get a cup of coffee. (Note that you are not required to drink it, just play with it.) Then slowly and carefully pour a little cream (or half and half, skim milk probably won't work) into it. Now ever so gently, give it a simple single stir with a stick or a spoon and observe the complex display that you see. Assuming that the cream and coffee move together, and that the vorticity (at least for a while) moves like fluid material, explain what you see in the light of Example 2 above.

3.6 The Turbulence Closure Problem and the Eddy Viscosity

From the point of view of the averaged motion, at least, the problem with the non-linearity of the instantaneous equations is that they introduce new unknowns, the Reynolds stress into the averaged equations. There are six individual stress components we must deal with to be exact: $< u_1^2 >$, $< u_2^2 >$, $< u_3^2 >$, $< u_1 u_2 >$, $< u_1 u_3 >$, and $< u_2 u_3 >$. These have to be related to the mean motion itself before the equations can be solved, since the number of unknowns and number

[5]Starbucks is a very popular chain of coffee shops in the USA and many other countries who have only recently discovered what good coffee tastes like.

of equations must be equal. The absence of these additional equations is often referred to as **the Turbulence Closure Problem**.

A similar problem arose when the instantaneous equations were written (equations 3.1 and 3.2), since relations had to be introduced to relate the stresses (in particular, the viscous stresses) to the motion itself. These relations (or constitutive equations) *depended only on the properties of the fluid material, and not on the flow itself*. Because of this fact, it is possible to carry out independent experiments, called viscometric experiments, in which these fluid properties can be determined once and for all. Equation 3.5 provides an example of just such a constitutive relation, the viscosity, μ, depending only in the choice of the material. For example, once the viscosity of water at given temperature is determined, this value can be used in all flows at that temperature, not just the one in which the evaluation was made.

It is tempting to try such an approach for the turbulence Reynolds stresses (even though we know the underlying requirements of scale separation are not satisfied). For example, a Newtonian type closure for the Reynolds stresses, often referred to as an "eddy" or "turbulent" viscosity model, looks like:

$$-\rho < u_i u_j >= \mu_t \left[S_{ij} - \frac{1}{3} S_{kk} \delta_{ij} \right] \tag{3.23}$$

where μ_t is the turbulence "viscosity" (also called the eddy viscosity), and S_{ij} is the *mean* strain rate defined by:

$$S_{ij} = \frac{1}{2} \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] \tag{3.24}$$

The second term, of course, vanishes identically for incompressible flow. For the simple case of a two-dimensional shear flow, equation 3.23 for the Reynolds shear stress reduces to

$$-\rho < u_1 u_2 >= \mu_t \frac{\partial U_1}{\partial x_2} \tag{3.25}$$

That such a simple model can adequately describe the mean motion in at least one flow is illustrated by the axisymmetric buoyant plume sketched in Figure 3.1. Figures 3.2 and 3.3 show the calculation of the mean velocity and temperature profiles respectively. Obviously the mean velocity and temperature profiles are reasonably accurately computed, as are the Reynolds shear stress and lateral turbulent heat flux shown in Figures 3.4 and 3.5.

The success of the eddy viscosity in the preceding example is more apparent than real, however, since the value of the eddy viscosity and eddy diffusivity (for the turbulent heat flux) have been chosen to give the best possible agreement with the data. This, in itself, would not be a problem if that chosen values could have been obtained in advance of the computation, or even if they could be used to successfully predict other flows. In fact, the values used work only for this flow, thus the computation is *not a prediction at all, but a* **postdiction** *or* **hindcast**

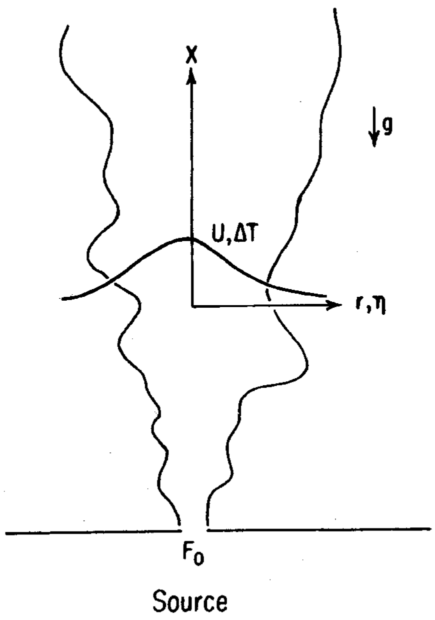

Figure 3.1: Plume

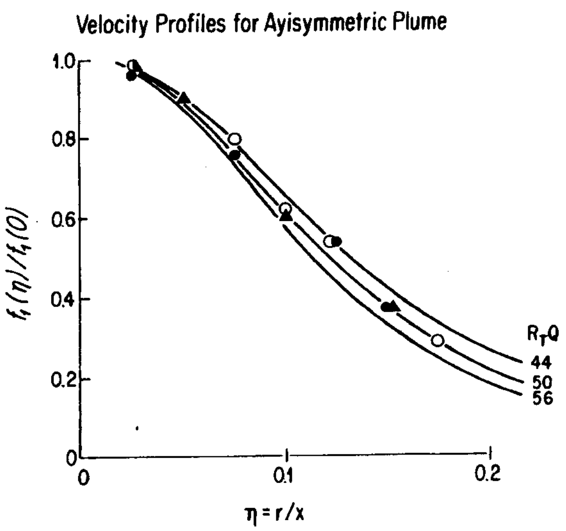

Figure 3.2: Velocity Profiles for Axisymmetric Plume

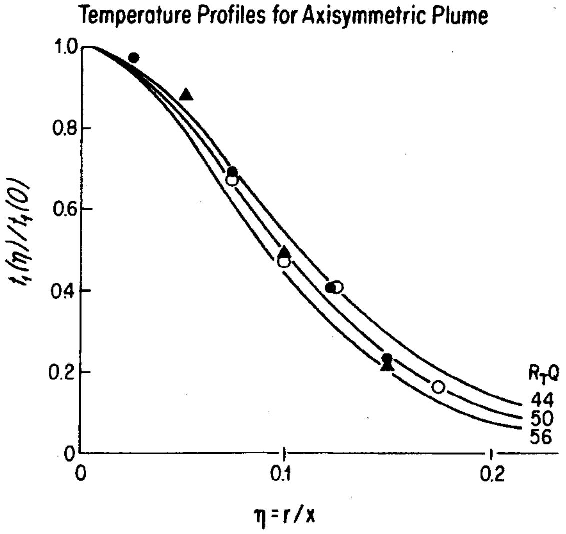

Figure 3.3: Temperature Profiles for Axisymmetric Plume

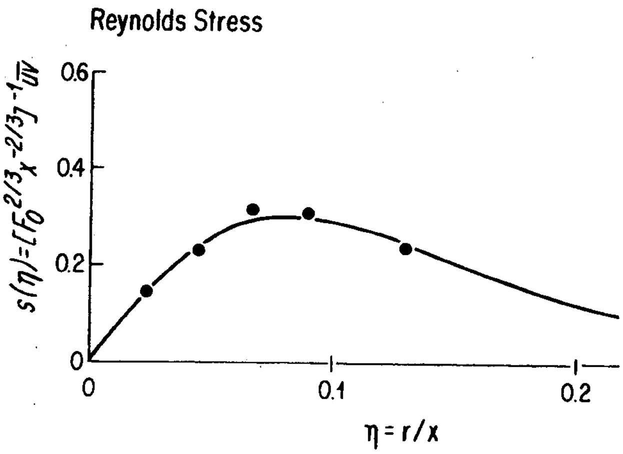

Figure 3.4: Reynolds Stress

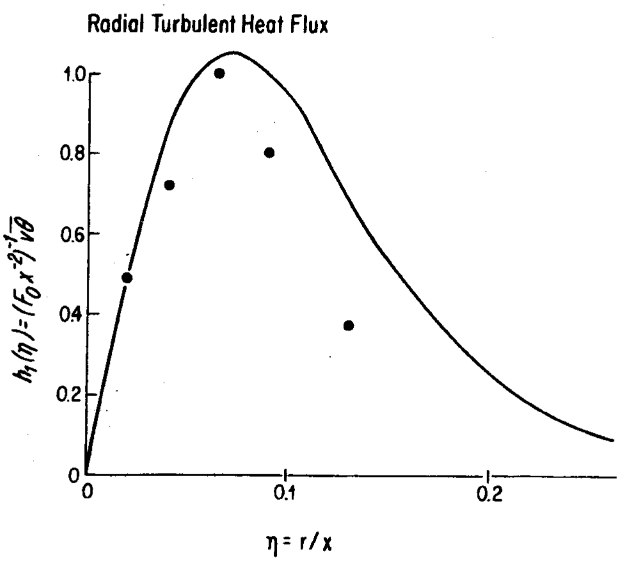

Figure 3.5: Radial Turbulence Heat Flux

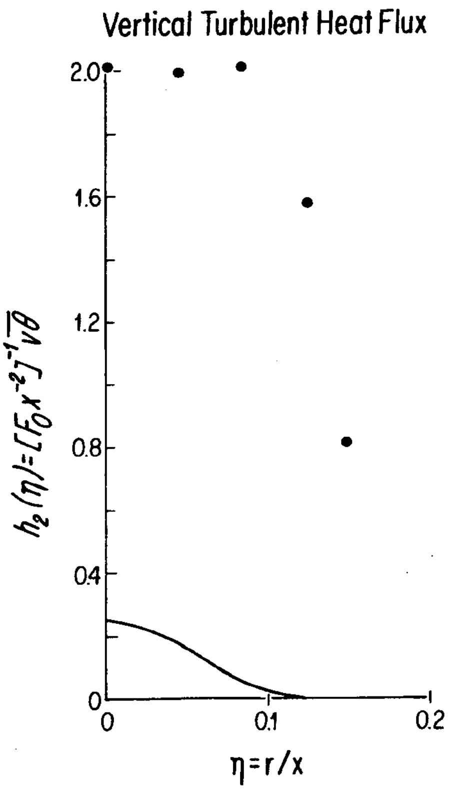

Figure 3.6: Vertical Turbulence Heat Flux

from which no extrapolation to the future can be made. In other words, our turbulence "model" is about as useful as having a program to predict yesterday's weather. Thus the closure problem still very much remains.

Another problem with the eddy viscosity in the example above is that it fails to calculate the vertical components of the Reynolds stress and turbulent heat flux. An attempt at such a computation is shown in Figure 3.6 where the vertical turbulent heat flux is shown to be severely underestimated. Clearly the value of the eddy viscosity in the vertical direction must be different than in the radial direction. In other words, the turbulence for which a constitutive equation is being written is *not an isotropic "medium"*. In fact, in this specific example the problem is that the vertical component of the heat flux is produced more by the interaction of buoyancy and the turbulence, than it is by the working of turbulence against mean gradients in the flow. We will discuss this in more detail in the next chapter when we consider the turbulence energy balances, but note for now that simple gradient closure models never work unless gradient production dominates. This rules out many flows involving buoyancy, and also many involving recirculations or separation where the local turbulence is convected in from somewhere else.

A more general form of constitutive equation which would allow for the non-isotropic nature of the "medium" (in this case the turbulence itself) would be

$$-\rho < u_i u_j >= \mu_{ijkl} \left[S_{kl} - \frac{1}{3} S_{mm} \delta_{kl} \right] \tag{3.26}$$

This closure relation allows each component of the Reynolds stress to have its own unique value of the eddy viscosity. It is easy to see that it is unlikely this will solve the closure problem since the original six unknowns, the $< u_i u_j >$, have been traded for eighty-one new ones, μ_{ijkl}. Even if some can be removed by symmetries, the remaining number is still formidable. More important than the number of unknowns, however, is that there is no independent or general means for selecting them without considering a particular flow. This is because *turbulence is indeed a property of the flow, not of the fluid.*

3.7 The Reynolds Stress Equations

It is clear from the preceding section that the simple idea of an eddy viscosity might not be the best way to approach the problem of relating the Reynolds stress to the mean motion. An alternative approach is to try to derive dynamical equations for the Reynolds stresses from the equations governing the fluctuations themselves. Such an approach recognizes that the Reynolds stress is really a functional of the velocity; that is, the stress at a point depends on the velocity everywhere and for all past times, not just at the point in question and at a particular instant in time.

The analysis begins with the equation for the instantaneous fluctuating velocity, equation 3.21. This can be rewritten for a Newtonian fluid with constant viscosity as:

$$\rho\left[\frac{\partial u_i}{\partial t}+U_j\frac{\partial u_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+\frac{\partial \tau^{(v)}_{ij}}{\partial x_j}-\rho\left[u_j\frac{\partial U_i}{\partial x_j}\right]-\rho\left\{u_j\frac{\partial u_i}{\partial x_j}-\left\langle u_j\frac{\partial u_i}{\partial x_j}\right\rangle\right\} \tag{3.27}$$

Note that the free index in this equation is i. Also, since we are now talking about turbulence again, the capital letters represent mean or averaged quantities.

Multiplying equation 3.27 by u_k and averaging yields:

$$\begin{aligned}\rho\left[\left\langle u_k\frac{\partial u_i}{\partial t}\right\rangle+U_j\left\langle u_k\frac{\partial u_i}{\partial x_j}\right\rangle\right] &= -\left\langle u_k\frac{\partial p}{\partial x_i}\right\rangle+\left\langle u_k\frac{\partial \tau^{(v)}_{ij}}{\partial x_j}\right\rangle \\ &\quad -\rho\left[< u_k u_j > \frac{\partial U_i}{\partial x_j}\right]-\rho\left\{\left\langle u_k u_j\frac{\partial u_i}{\partial x_j}\right\rangle\right\}\end{aligned} \tag{3.28}$$

Now since both i and k are free indices they can be interchanged to yield a second equation given by[6]:

$$\begin{aligned}\rho\left[\left\langle u_i\frac{\partial u_k}{\partial t}\right\rangle+U_j\left\langle u_i\frac{\partial u_k}{\partial x_j}\right\rangle\right] &= -\left\langle u_i\frac{\partial p}{\partial x_k}\right\rangle+\left\langle u_i\frac{\partial \tau^{(v)}_{kj}}{\partial x_j}\right\rangle \\ &\quad -\rho\left[< u_i u_j > \frac{\partial U_k}{\partial x_j}\right]-\rho\left\{\left\langle u_i u_j\frac{\partial u_k}{\partial x_j}\right\rangle\right\}\end{aligned} \tag{3.29}$$

Equations 3.28 and 3.29 can be added together to yield an equation for the Reynolds stress,

$$\begin{aligned}\frac{\partial < u_i u_k >}{\partial t}+U_j\frac{\partial < u_i u_k >}{\partial x_j} &= -\frac{1}{\rho}\left[\left\langle u_i\frac{\partial p}{\partial x_k}\right\rangle+\left\langle u_k\frac{\partial p}{\partial x_i}\right\rangle\right] \\ &\quad -\left[\left\langle u_i u_j\frac{\partial u_k}{\partial x_j}\right\rangle+\left\langle u_k u_j\frac{\partial u_i}{\partial x_j}\right\rangle\right] \\ &\quad +\frac{1}{\rho}\left[\left\langle u_i\frac{\partial \tau^{(v)}_{kj}}{\partial x_j}\right\rangle+\left\langle u_k\frac{\partial \tau^{(v)}_{ij}}{\partial x_j}\right\rangle\right] \\ &\quad -\left[< u_i u_j > \frac{\partial U_k}{\partial x_j}+< u_k u_j > \frac{\partial U_i}{\partial x_j}\right]\end{aligned} \tag{3.30}$$

It is customary to rearrange the first term on the right hand side in the following way:

$$\begin{aligned}\left[\left\langle u_i\frac{\partial p}{\partial x_k}\right\rangle+\left\langle u_k\frac{\partial p}{\partial x_i}\right\rangle\right] &= \left\langle p\left[\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right]\right\rangle \\ &\quad -\frac{\partial}{\partial x_j}\left[< p u_i > \delta_{kj}+< p u_k > \delta_{ij}\right]\end{aligned} \tag{3.31}$$

[6] Alternatively equation 3.21 can be rewritten with free index k, then multiplied by u_i and averaged

The first term on the right is generally referred to as the *pressure strain-rate* term. The second term is written as a divergence term, and is generally referred to as the *pressure diffusion* term. We shall see later that divergence terms can never create nor destroy anything; they can simply move it around from one place to another.

The third term on the right-hand side of equation 3.30 can similarly be rewritten as:

$$\begin{aligned}\left[\left\langle u_i \frac{\partial \tau_{kj}^{(v)}}{\partial x_j}\right\rangle + \left\langle u_k \frac{\partial \tau_{ij}^{(v)}}{\partial x_j}\right\rangle\right] &= - \left[\left\langle \tau_{ij}^{(v)} \frac{\partial u_k}{\partial x_j}\right\rangle + \left\langle \tau_{kj}^{(v)} \frac{\partial u_i}{\partial x_j}\right\rangle\right] \\ &+ \frac{\partial}{\partial x_j}[< u_i \tau_{kj}^{(v)} > + < u_k \tau_{ij}^{(v)} >]\end{aligned} \tag{3.32}$$

The first of these is also a divergence term. For a Newtonian fluid, the last is the so-called "dissipation of Reynolds stress" by the turbulence viscous stresses. This is easily seen by substituting the Newtonian constitutive relation to obtain:

$$\left[\tau_{ij}^{(v)} \frac{\partial u_k}{\partial x_j} + \tau_{kj}^{(v)} \frac{\partial u_i}{\partial x_j}\right] = 2\nu \left[\left\langle s_{ij} \frac{\partial u_k}{\partial x_j}\right\rangle + \left\langle s_{kj} \frac{\partial u_i}{\partial x_j}\right\rangle\right] \tag{3.33}$$

It is not at all obvious what this has to do with dissipation, but it will become clear later on when we consider the trace of the Reynolds stress equation, which is the *kinetic energy* equation for the turbulence.

Now if we use the same trick from before using the continuity equation, we can rewrite the third term in equation 3.30 to obtain:

$$\left[\left\langle u_i u_j \frac{\partial u_k}{\partial x_j}\right\rangle + \left\langle u_k u_j \frac{\partial u_i}{\partial x_j}\right\rangle\right] = \frac{\partial}{\partial x_j} < u_i u_k u_j > \tag{3.34}$$

This is also a divergence term.

We can use all of the pieces we have developed above to rewrite equation 3.30 as:

$$\begin{aligned}\frac{\partial}{\partial t} < u_i u_k > &+ U_j \frac{\partial}{\partial x_j} < u_i u_k >= - \left\langle \frac{p}{\rho}\left[\frac{\partial u_i}{\partial x_k} + \frac{\partial u_i}{\partial x_k}\right]\right\rangle \\ &+ \frac{\partial}{\partial x_j}\{-[< p u_k > \delta_{ij} + < p u_i > \delta_{kj}] - < u_i u_k u_j > \\ &\qquad + 2\nu[< s_{ij} u_k > + < s_{kj} u_i >]\} \\ &- \left[< u_i u_j > \frac{\partial U_k}{\partial x_j} + < u_k u_j > \frac{\partial U_i}{\partial x_j}\right] \\ &- 2\nu \left[\left\langle s_{ij} \frac{\partial u_k}{\partial x_j}\right\rangle + \left\langle s_{kj} \frac{\partial u_i}{\partial x_j}\right\rangle\right]\end{aligned} \tag{3.35}$$

This is the so-called **Reynolds Stress Equation** which has been the primary vehicle for much of the turbulence modeling efforts of the past few decades.

The left hand side of the Reynolds Stress Equation can easily be recognized as the rate of change of Reynolds stress following the mean motion. It seems to provide exactly what we need: nine new equations for the nine unknowns we cannot account for. The problems are all on the right-hand side. These terms are referred to respectively as

1. the pressure-strain rate term
2. the turbulence transport (or divergence) term
3. the "production" term, and
4. the "dissipation" term.

Obviously these equations do not involve only U_i and $< u_i u_j >$, but depend on many more new unknowns.

It is clear that, contrary to our hopes, we have not derived a single equation relating the Reynolds stress to the mean motion. Instead, our Reynolds stress transport equation is exceedingly complex. Whereas the process of averaging the equation for the mean motion introduced only six new, independent unknowns, the Reynolds stress, $< u_i u_j >$, the search for a transport equation which will relate these to the mean motion has produced many more unknowns. They are:

$$< pu_i > \quad - \quad 3 \quad unknowns \tag{3.36}$$
$$< u_i s_{jk} > \quad - \quad 27 \tag{3.37}$$
$$< s_{ij} s_{jk} > \quad - \quad 9 \tag{3.38}$$
$$< u_i u_k u_j > \quad - \quad 27 \tag{3.39}$$
$$< p \frac{\partial u_i}{\partial x_j} > \quad - \quad 9 \tag{3.40}$$
$$TOTAL \quad - \quad 75 \tag{3.41}$$

Not all of these are independent, since some can be derived from the others. Even so, our goal of reducing the number of unknowns has clearly not been met.

Equations governing each of these new quantities can be derived from the original dynamical equations, just as we did for the Reynolds stress. Unfortunately new quantities continue to be introduced with each new equation, and at a faster rate than the increase in the number of equations. Now the full implications of the closure problem introduced by the Reynolds decomposition and averaging has become apparent. No matter how many new equations are derived, the number of new unknown quantities introduced will always increase more rapidly.

Our attempt to solve the turbulence problem by considering averages illustrates a general principle. Any time we try to fool Mother Nature by averaging out her details, she gets her revenge by leaving us with a closure problem — more equations than unknowns. In thermodynamics, we tried to simplify the consideration of molecules by averaging over them, and were left with the need for an equation of state. In heat transfer, we tried to simplify considerations by which molecules transfer their kinetic energy, and found we were lacking a relation between the heat flux and the temperature field. And in fluid mechanics, we tried to simplify consideration of the "mean" motion of molecules and ended up with viscous stress. In all of these cases we were able to make simple physical models which worked at least some of the time; e.g., ideal gas, Fourier-Newtonian fluid. And these models all worked because we were able to make assumptions about the underlying molecular processes and assume them to be independent of the macroscopic flows of interest. Unfortunately such assumptions are rarely satisfied in turbulence.

It should be obvious by now that the turbulence closure problem will not be solved by the straight-forward derivation of new equations, nor by direct analogy with viscous stresses. Rather, *closure attempts will have to depend on an intimate knowledge of the dynamics of the turbulence itself.* Only by understanding how the turbulence behaves can one hope to *guess* an appropriate set of constitutive equations **AND** *understand the limits of them.* This is, of course, another consequence of the fact that the *turbulence is a property of the flow itself, and not of the fluid*!