Partial Differential Equations and Complex Variable

Notes by H C Rae

Department of Mathematics King's College London

Contents

Chapter 1

Introduction

1.1 Why is this course important?

CM211A is primarily a *mathematical methods* course and depends heavily on the the results obtained in CM112A (Calculus II). It aims to provide students with a basic knowledge of Partial Differential Equations (PDE) and Complex Variable theory.

The section on Complex Variable aims to introduce the student to the fundamentals of the theory of functions of a complex variable without undue emphasis on rigour. In some areas where rigour is lacking the defects will be made good in the Third Year course CM322C (Complex Analysis). Complex Variable theory is of great importance in both pure and applied mathematics. On the applied side the applications range from electrical engineering and fluid mechanics to the theory of elementary particles. We shall see that the real and imaginary parts of an analytic function satisfy the two-dimensional Laplace equation and this result provides a link between the two strands of the course.

In the PDE section we will consider various PDEs, especially some of the partial differential equations of Mathematical Physics, the two-dimensional Laplace equation being an example. As we study some of these equations the student will be introduced to several techniques which are of fundamental importance in applied mathematics. For example, the method of separation of variables applied to the one-dimensional wave equation soon leads to the study of Fourier series and Fourier transforms, both of which are of great importance in many areas of mathematics, both pure and applied.

1.2 How to succeed in this course

The material covered in CM211A is extremely important for all mathematics students. Although it is *not* an exceptionally difficult course, in the sense of being abstract or conceptually demanding, it contains a lot of important material, and in order to succeed you should take serious note of the following comments.

First of all, you should attend all the lectures and tutorial classes. Don't be deceived: You have a set of course notes, but this is not in itself a guarantee of success! The notes have been produced so that you can attend lectures without the need to write a lot during them. Attendance at lectures is very important, as is attendance at tutorial classes where problems are discussed.

Second, you must try the weekly problems. Failure to make a serious attempt at these problems is one of the main reasons for failure in this course. You may struggle with some of the problems, but wrestling with a problem yourself (even if you don't succeed) is very beneficial in developing your understanding. Remember this: It is a delusion to think that you can learn a subject solely by watching someone else write out solutions to problems on a blackboard — one learns by personal involvement. It is also a delusion to think that possessing a set of solutions to problems and a set of printed lecture notes somehow absolves you from the need to use pen and paper yourself! It is a sad fact that the simplest question can be difficult for a student whose first attempt at doing a problem is when he/she is seated at a desk in the Royal Horticultural Halls in the month of June. So please take note of what I've said now, and don't learn the hard way!

If you are in difficulty, talking to other students, consulting books, or asking for guidance from a lecturer or tutor will help. Lecturers aren't spiteful or vengeful people, but for those who ignore sound advice there is a ring of truth in the following quotation from the book of Proverbs[1]:

> Because I have called, and ye refused; I have stretched out my hand, and no man regarded;
>
> But ye have set at nought all my counsel, and would have none of my reproof:
>
> I also will laugh at your calamity: I will mock when your fear cometh ......

I've tried to be careful in typing up these notes but it's inevitable that there will be typing errors; if you find any, please let me know so that I can eliminate them from any subsequent editions. Thank you.

[1] Proverbs chapter 1, King James translation of the Bible

Chapter 2

Revision of Complex Numbers

2.1 Notation

For a typical complex number $z \in \mathbf{C}$ we write $z = x + iy,\ x, y \in \mathbf{R}$. In the polar representation $x = r\cos\theta,\ y = r\sin\theta$ so that $z = r(\cos\theta + i\sin\theta)$. Here θ is the argument of z, written $\theta = \arg(z)$. $\arg(z)$ is multi-valued in the sense that if θ_0 is a permitted value of $\arg(z)$ so is $\theta_0 + 2k\pi$ where $k = 0, \pm1, \pm2, \ldots$. This is just a reflection of the fact that the sine and cosine functions are periodic with period 2π. The value of $\arg(z)$ which satisfies the inequality $-\pi < \arg(z) \leq \pi$ is called the *principal value* of $\arg(z)$, sometimes denoted by $\mathrm{Arg}(z)$. Thus, if $z = i$ $\mathrm{Arg}(z) = \pi/2$; if $z = -i$ $\mathrm{Arg}(z) = -\pi/2$; if $z = -1$ $\mathrm{Arg}(z) = \pi$. r is the *modulus* of z, written $r = |z|$.

We note that

$$\tan(\arg(z)) = \frac{y}{x},\ (x \neq 0), \qquad r = |z| = \sqrt{(x^2 + y^2)}.$$

The first of these formulae does not necessarily imply that $\arg(z) = \arctan(y/x)$ since arctan has values in $(-\pi/2, \pi/2)$. For example, if $z = -1 - i$ then $\tan\arg(z) = (-1)/(-1) = 1$ and $\mathrm{Arg}(z) = -3\pi/4$ whereas $\arctan(1) = \pi/4$.

Note that $r = \sqrt{(x^2 + y^2)}$ — *not* $r = \sqrt{(x^2 - y^2)}$!

We note the following

- As a matter of definition we put $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$. The definition is justified by the fact that $\mathrm{e}^{i\theta_1}\mathrm{e}^{i\theta_2} = \mathrm{e}^{i(\theta_1+\theta_2)}$ (a property which is readily verified using standard identities for the sine and cosine functions). In particular $\mathrm{e}^{i\theta}\mathrm{e}^{-i\theta} = 1$, $\mathrm{e}^{-i\theta} = 1/\mathrm{e}^{i\theta},\ (\mathrm{e}^{i\theta})^n = \mathrm{e}^{ni\theta}$, for any integer n.
- For any $z_1, z_2 \in \mathbf{C}$ $|z_1 z_2| = |z_1||z_2|$, from which it follows that $|z_1/z_2| = |z_1|/|z_2|$ if $z_2 \neq 0$.
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$

- The complex conjugate of $z = x+iy$ we denote by $\overline{z}$ so that $\overline{z} = x-iy$. (some books use z^* instead of $\overline{z}$). The following simple results hold:

$$z\overline{z} = x^2 + y^2 = |z|^2, \quad |\overline{z}| = |z|, \quad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 z_2} = \overline{z}_1\overline{z}_2.$$

- In order to rotate a complex number z about 0 through an angle α we multiply it by $\mathrm{e}^{i\alpha}$. For, writing z in polar form, $z = |z|\mathrm{e}^{i\theta}$, $z\mathrm{e}^{i\alpha} = |z|\mathrm{e}^{i(\theta+\alpha)}$; the complex number on the righthand side of this equation has modulus equal to $|z|$ and argument equal to $\theta+\alpha$. (We use the fact that for any real γ $|\mathrm{e}^{i\gamma}| = 1$, a property which we use time and time again). In particular we rotate z through an angle of $\pi/2$ by multiplying it by $\mathrm{e}^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$. In order to rotate z through an angle of $\pi/4$ we multiply it by $\mathrm{e}^{i\pi/4} = \cos\pi/4 + i\sin\pi/4 = (1+i)/\sqrt{2}$, and so on.

Example 2.1 *Express* $\cos 5\theta$ *in powers of* $\cos\theta$.

$$\begin{aligned}
(\cos 5\theta + i\sin 5\theta) &= \mathrm{e}^{5i\theta} = (\mathrm{e}^{i\theta})^5 = (\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 \\
&+ 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5
\end{aligned}$$

Taking the real part of this eqation and writing $c = \cos\theta,\ s = \sin\theta$ *we obtain*

$$\begin{aligned}
\cos 5\theta &= c^5 - 10c^3s^2 + 5cs^4 = c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2 \\
&= 16c^5 - 20c^3 + 5c
\end{aligned}$$

Setting $\theta = \pi/10$ *in this formula gives*

$$0 = 16C^5 - 20C^3 + 5C \quad (C = \cos\pi/10), \quad 16C^4 - 20C^2 + 5 = 0$$

from which we derive

$$C^2 = \frac{20 \pm 4\sqrt{5}}{32} = \frac{5 \pm \sqrt{5}}{8}.$$

It is easy to argue that the positive sign is appropriate and consequently

$$\cos\pi/10 = \left(\frac{5+\sqrt{5}}{8}\right)^{1/2}.$$

2.2 Roots of unity

Let n be a positive integer. We want to find the n roots in $\mathbf{C}$ of the equation $z^n = 1$. To this end write $z = r\mathrm{e}^{i\theta}$, $r = |z|$, $\theta = arg(z)$. Substitution gives $r^n(\mathrm{e}^{i\theta})^n = 1$ and therefore $r^n\mathrm{e}^{ni\theta} = (1)\mathrm{e}^{i(0+2k\pi)}$, where k is any integer. It follows that $r^n = 1$, so that $r = 1$ (since r is real), and that $n\theta = 2k\pi$. We can choose $k = 0, 1, 2, 3, \ldots (n-1)$ to generate the required n roots; other values of k merely give repetitions, as one easily checks. We conclude that the n solutions of $z^n = 1$ are given by

$$z = z_k = \mathrm{e}^{i(2k\pi/n)}, \quad k = 0, 1, 2, 3, \ldots, (n-1).$$

The n values z_k, $k = 0, 1, 2, 3, \ldots, (n-1)$ are referred to as the n-th roots of unity. $k = 0$ gives the obvious real root $z = z_0 = 1$.

Geometrically the n-th roots of unity lie on the unit circle centre 0; the angular separation between consecutive roots is clearly $2\pi/n$. As an illustration consider the following example.

Example 2.2 *The roots of the equation $z^7 = 1$ are given by*

$$z = z_k = \mathrm{e}^{i(2k\pi/7)} = \cos(2k\pi/7) + i\sin(2k\pi/7), k = 0, \pm1, \pm2, \pm3.$$

(It's convenient to choose these values of k rather than $k = 0, 1, 2, 3, 4, 5, 6.$ to generate the 7 roots) Now

$$(z^7 - 1) = (z-1)(z-z_1)(z-z_{-1})(z-z_2)(z-z_{-2})(z-z_3)(z-z_{-3}).$$

Since $(z-z_k)(z-z_{-k}) = z^2 - z(z_k + z_{-k}) + z_k z_{-k} = z^2 - 2z\cos(2k\pi/7) + 1$ we deduce that

$$\frac{z^7-1}{z-1} = (z^2 - 2z\cos(2\pi/7)+1)(z^2 - 2z\cos(4\pi/7)+1)(z^2 - 2z\cos(6\pi/7)+1), \ z \neq 1.$$

The left-hand side is equal to $1 + z + z^2 + \cdots + z^6$ and if we now let $z \to 1$ we obtain the formula

$$7 = 2^3(1-\cos(2\pi/7))(1-\cos(4\pi/7))(1-\cos(6\pi/7)) = 2^3 2^3 \sin^2(\pi/7)\sin^2(2\pi/7)\sin^2(3\pi/7).$$

Equivalently

$$\sin^2(\pi/7)\sin^2(2\pi/7)\sin^2(3\pi/7) = 7/64.$$

This result can obviously be generalized by applying the same considerations to the equation $z^{(2n+1)} = 1$.

As a final example consider the following.

Example 2.3 *Express $\cos^6\theta$ as a linear combination of cosines of multiples of θ.*

Since $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$, $\mathrm{e}^{-i\theta} = \cos\theta - i\sin\theta$ we have the formulae

$$\cos\theta = \frac{1}{2}(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}), \qquad \sin\theta = \frac{1}{2i}(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}) \tag{2.1}$$

It follows that

$$\begin{aligned}
\cos^6\theta &= \frac{1}{2^6}(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta})^6 \\
&= \frac{1}{2^6}\Big(\mathrm{e}^{6i\theta} + 6\mathrm{e}^{5i\theta}\mathrm{e}^{-i\theta} + 15\mathrm{e}^{4i\theta}\mathrm{e}^{-2i\theta} + 20\mathrm{e}^{3i\theta}\mathrm{e}^{-3i\theta} + 15\mathrm{e}^{2i\theta}\mathrm{e}^{-4i\theta} + 6\mathrm{e}^{i\theta}\mathrm{e}^{-5i\theta} + \mathrm{e}^{-6i\theta}\Big)
\end{aligned}$$

Now $e^{ni\theta} + e^{-ni\theta} = 2\cos n\theta$ *by equation 2.1. We therefore obtain*

$$\cos^6\theta = \frac{1}{2^5}\Big(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\Big)$$

which expresses $\cos^6\theta$ *as a linear combination of* $1, \cos 2\theta, \cos 4\theta, \cos 6\theta$. *This result generalizes in an obvious way.*

Chapter 3

Functions of a Complex Variable

3.1 Introduction

In the following sections we shall begin our study of analytic functions of a complex variable. Complex variable theory is one of the most beautiful branches of pure mathematics but it also has important applications in applied mathematics, in the study of fluid mechanics in particular. The link with PDEs occurs through the two-dimensional Laplace equation.

In what follows we shall attempt to convey some of the basic ideas of complex variable theory without too much emphasis on rigour; a more rigorous account is presented in the third year course on complex analysis.

3.2 Sets in the complex plane

We make the following definitions:

Definition 3.1 *The open disc centre* $z_0 \in \mathbf{C}$ *and radius* $r > 0$ *is the set* $N_r(z_0)$ *given by*

$$N_r(z_0) = \{z \in \mathbf{C} : |z - z_0| < r\}.$$

Definition 3.2 *The closed disc centre* $z_0 \in \mathbf{C}$ *and radius* $r > 0$ *is the set* $\overline{N_r}(z_0)$ *given by*

$$\overline{N_r}(z_0) = \{z \in \mathbf{C} : |z - z_0| \leq r\}.$$

We may picture $\overline{N_r}(z_0)$ geometrically as the union of the open disc $N_r(z_0)$ and the circumference of the circle centre z_0 and radius r.

Definition 3.3 *A set* $G \subseteq \mathbf{C}$ *is open* $\iff$ *for any* $\zeta \in G$ $\exists r > 0$ *such that* $N_r(\zeta) \subseteq G$.

- $N_r(z_0)$ is an open set.

- $\mathbf{C}$ is open.
- $\{z : 0 < \Im z < 1\}$ is an open set.
- $\{z : 0 \leq \Im z < 1\}$ is *not* an open set.
- $\overline{N_r}(z_0)$ is *not* an open set.

We have not offered formal proofs of these claims but they should be intuitively clear, bearing in mind our definition of the term *open* as applied to sets in the complex plane.

Definition 3.4 *The line segment from z_0 to z_1 is the set $\overbrace{z_0\ z_1}$ given by*

$$\overbrace{z_0\ z_1} = \{z : z = z_0 + t(z_1 - z_0),\ 0 \leq t \leq 1\}$$

Definition 3.5 *A polygon in $\mathbf{C}$ is a set of the form*

$$\overbrace{z_0\ z_1} \cup \overbrace{z_1\ z_2} \cup \overbrace{z_2\ z_3} \cup \cdots \cup \overbrace{z_{n-1}\ z_n}$$

for some points $z_0, z_1, z_2, \ldots, z_{n-1}, z_n$.

Definition 3.6 *An open set G is said to be connected (polygonally) $\iff$ for any two points $z', z'' \in G$ there is a polygon lying entirely in G with end points z', z''.*

An open connected set D is called a *domain*.

Definition 3.7 *A domain D is convex if $z, \zeta \in D \Rightarrow \overbrace{z\ \zeta} \subseteq D$.*

You should aim to have a clear intuitive understanding of all the terms defined above.

3.3 Functions of a complex variable

A complex function f is a mappping f from a subset of $\mathbf{C}$ into $\mathbf{C}$. f is real valued if its range is a subset of $\mathbf{R}$.

We shall assume that f is defined on some domain D unless otherwise stated. Recall that a domain is an open set. This means that for any point $z_0 \in D$ we can find an open disc with centre z_0 and positive radius which lies inside D. This is important because it enables us to define concepts such as continuity and differentiability of f at z_0.

Definition 3.8 *A function $f : D \to \mathbf{C}$ is continuous at $z_0 \iff f(z) \to f(z_0)$ as $z \to z_0$.*

In terms of ϵ, δ language this means that given $\epsilon > 0$ $\exists \delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all z such that $|z - z_0| < \delta$.

Definition 3.9 *Let D be a domain and let $f : D \to \mathbf{C}$. We say that f is differentiable at $z_0 \in D \iff$*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists as a complex number $\alpha + i\beta$. More explicitly this means that f is differentiable at $z_0 \iff$ given $\epsilon > 0$ $\exists \delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - (\alpha + i\beta) \right| < \epsilon$$

for all z such that $0 < |z - z_0| < \delta$. If f is differentiable at z_0 we use the notations familiar from calculus and write $f'(z_0)$ or $\frac{df}{dz}(z_0)$ for the complex number $\alpha + i\beta$.

Differentiability is a very strong condition. Roughly speaking it means that $(f(z) - f(z_0))/(z - z_0)$ tends to the same value, however we allow z to approach z_0. The following example is very instructive and may help to illustrate how our intuition could go wrong.

Example 3.1 *For $z = x + iy \in \mathbf{C}$, define $f(z)$ by*

$$f(z) = \frac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0, \quad f(0) = 0.$$

Show that

$$\frac{f(z) - f(0)}{z} \to 0$$

as $z \to 0$ along any straight line through the origin, but prove that f is not differentiable at the origin by examining what happens as $z \to 0$ along the curve $x = y^2$.

We have:

$$\frac{f(z) - f(0)}{z} = \frac{xy^2(x + iy)}{(x^2 + y^4)(x + iy)} = \frac{xy^2}{x^2 + y^4}.$$

If we approach 0 along the lines $x = 0$ it's obvious that

$$\frac{f(z) - f(0)}{z} \to 0.$$

If we approach 0 along other radial directions we may write $y = \lambda x, \quad \lambda \in \mathbf{R}$. In these cases

$$\frac{f(z) - f(0)}{z} = \frac{x^3\lambda^2}{x^2 + \lambda^4 x^4} = \frac{x\lambda^2}{1 + \lambda^4 x^2} \to 0 \text{ as } z \to 0 \ \forall \lambda \in \mathbf{R}.$$

However, if we approach along the curve Γ whose equation is $x = y^2$ (every open disc centre 0 contains points of Γ) then

$$\frac{f(z) - f(0)}{z} = \frac{y^4}{y^4 + y^4} = \frac{1}{2} \to \frac{1}{2} \text{ as } z \to 0 \text{ along } \Gamma.$$

It follows that f is *not* differentiable at $z = 0$.

One can prove the differentiability of many functions by the standard methods of real variable calculus. For example, if $f(z) = z^2$ we have

$$\lim_{z \to z_0} \frac{z^2 - {z_0}^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0.$$

f is differentiable at any point z_0 and $f'(z_0) = 2z_0$.

The standard rules of differentiation apply. For example, if f, g are differentiable at z, and λ is an arbitary complex constant, then $(f+g),\ \lambda f,\ fg,\ f/g$ are all differentiable at z (except that in the case of f/g we have to postulate that $g(z) \neq 0$). Moreover,

$$(f+g)' = f' + g',\ (\lambda f)' = \lambda f',\ (fg)' = fg' + f'g,\ (f/g)' = (gf' - fg')/(g^2).$$

(In these formulae we have omitted reference to the argument z)

It is not hard to prove that $\frac{d}{dz} z^n = nz^{n-1}$, where n is any integer — except that if n is negative we must exclude the point $z = 0$.

As for functions of a real variable, differentiability implies continuity. For, let f be differentiable at z_0. Then

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{(z - z_0)}(z - z_0) \to f'(z_0) \times 0 = 0$$

as $z \to z_0$. This shows that $f(z) \to f(z_0)$ as $z \to z_0$; in other words, f is continuous at z_0.

Before proceeding we illustrate the definition of differentiability by two more examples.

Example 3.2 *Let $f(z) = \overline{z},\ z \in \mathbf{C}$. For $z_0 \in \mathbf{C}$ we have, writing $z = z_0 + \omega$,*

$$\frac{(f(z_0 + \omega) - f(z_0))}{(z_0 + \omega) - z_0} = \frac{\overline{z_0 + \omega} - \overline{z_0}}{\omega} = \frac{\overline{\omega}}{\omega}.$$

Writing $\omega = \lambda + i\mu$ we need to consider the limit of $\overline{\omega}/\omega$ as $\omega \to 0$.
Choosing $\mu = 0,\ \lambda \neq 0,\ \overline{\omega}/\omega = \lambda/\lambda = 1 \to 1$ as $\lambda \to 0$.
On the other hand, with $\lambda = 0,\ \mu \neq 0,\ \overline{\omega}/\omega = \frac{-i\mu}{i\mu} = -1 \to -1$ as $\mu \to 0$.

It follows that

$$\lim_{z \to z_0} \frac{(f(z) - f(z_0))}{z - z_0}$$

does not exist and consequently f is not differentiable at any point z_0.

Example 3.3 *Let* $f(z) = |z|^2,\ z \in \mathbf{C}$. *Bearing in mind the basic formula* $\gamma\overline{\gamma} = |\gamma|^2$ *we have*

$$\frac{f(z+\omega)-f(z)}{\omega} = \frac{(z+\omega)(\overline{z}+\overline{\omega})-z\overline{z}}{\omega} = \frac{\omega\overline{z}+z\overline{\omega}+\omega\overline{\omega}}{\omega} = \overline{z} + z\frac{\overline{\omega}}{\omega} + \overline{\omega}.$$

Whether f *is differentiable at the point* z *or not depends on whether the righthand side of this equation has a limit as* $\omega \to 0$. *It is clear that* $\overline{\omega} \to 0$ *as* $\omega \to 0$ *but this is not true of* $\overline{\omega}/\omega$ *as we saw in the previous example (see the argument used in example 3.2). However, in the special case* $z = 0$ *the term* $\overline{\omega}/\omega$ *disappears and the righthand side tends to* $\overline{z} = 0$ *(in the case* $z = 0$*). Our final conclusion is therefore that* f *is differentiable at* $z = 0$, *where* $f'(0) = 0$, *but at no other point.*

3.4 Analytic functions

Definition 3.10 *Let* D *be a domain and* $f : D \to \mathbf{C}$. *We say that* f *is analytic at* $z_0 \in D \iff \exists \rho > 0$ *such that* f *is differentiable at each point of* $N_\rho(z_0)$. *If* f *is analytic at each point of* D *we say that* f *is analytic in* D. *A function which is analytic in* $\mathbf{C}$ *is said to be entire.*

(The terms *regular* or *holomorphic* are often used instead of analytic)

Theorem 3.1 *Suppose that* f *is analytic in a domain* D *and that* $f(z) = u(x,y) + iv(x,y),\ z = x + iy$. *Then the partial derivatives* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ *exist and satisfy the Cauchy*[1]*-Riemann*[2] *equations for* u, v *with respect to the variables (x,y)*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.1}$$

In order to prove this result it's convenient to use traditional notation and write $\delta z = \delta x + i\delta y$. Since f is analytic

$$\frac{f(z+\delta z)-f(z)}{\delta z} \to f'(z) \text{ as } \delta z \to 0,\ z \in D.$$

Now

$$\frac{f(z+\delta z)-f(z)}{\delta z} = \frac{u(x+\delta x, y+\delta y)-u(x,y)}{\delta z} + i\frac{v(x+\delta x, y+\delta y)-v(x,y)}{\delta z}.$$

Choosing $\delta y = 0$, so that $\delta z = \delta x$, and letting δz tend to zero we deduce that $\frac{\partial u}{\partial x}$

[1]Baron Augustin Louis Cauchy (1789-1857) French mathematician noted for his work on the theory of functions and the wave theory of light

[2]Georg Friedrich Bernhard Riemann (1826-1866) German mathematician famous for his work on analysis and non-Euclidean geometry

exists, that $\frac{\partial v}{\partial x}$ exists, and that

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (3.2)$$

In a similar way, this time choosing $\delta x = 0$, so that $\delta z = i\delta y$, we see that

$$\frac{f(z+\delta z) - f(z)}{\delta z} = -i\frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \frac{v(x, y+\delta y) - v(x,y)}{\delta y}.$$

Proceeding to the limit as δz tends to zero we deduce that $\frac{\partial u}{\partial y}$ exists, that $\frac{\partial v}{\partial y}$ exists and that

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad (3.3)$$

Comparing equations 3.2 and 3.3 we conclude that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

as required.

We state without proof a partial converse of this result.

Theorem 3.2 *Suppose that $f : D \to \mathbf{C}$, where D is a domain, and that $f(z) = u(x,y) + iv(x,y)$, $z = x + iy \in D$. Suppose that each of the partial derivatives u_x, u_y, v_x, v_y (we use subscript notation to make the text more readable at this point) exists and is continuous on D and that u, v satisfy the Cauchy-Riemann equations in D. Then f is analytic in D.*

It should be clear from our work on differentiability that if f, g are analytic in a domain D, and $\lambda \in \mathbf{C}$ is constant, then so are $(f+g)$, λf, fg; f/g is also analytic, always supposing that g has no zeros in D.

We also note that an analytic function of an analytic function is analytic and the usual chain rule of differentiation applies. Thus, if u is an analytic function of ζ, $u = f_1(\zeta)$, and ζ is an analytic function of z, $\zeta = f_2(z)$, then $u = f_1(f_2(z))$ is an analytic function of z and

$$\frac{du}{dz} = \frac{du}{d\zeta}\frac{d\zeta}{dz}.$$

Theorem 3.3 *Suppose that f is analytic in a domain D and that $f'(z) = 0$ at all points of D. Then f is constant on D.*

Let P_0, with affix z_0, be any point of D. Given a point P with affix ζ in D, we can connect P_0 to P by a polygon (since a domain is polygonally connected) which we may denote by

$$\overbrace{P_0\ P_1} \cup \overbrace{P_1\ P_2} \cup \overbrace{P_2\ P_3} \ldots \cup \overbrace{P_{n-1}\ P}$$

Our theorem will be established if we can show that f is constant on the individual line segments. For this purpose it will suffice to consider the line segment $\overbrace{P_0\ P_1}$. Let z_1 be the affix of P_1. A typical point on this line segment has affix $z(t) = z_0 + t(z_1 - z_0)$, $t \in [0, 1]$. (P_0 corresponds to $t = 0$, P_1 corresponds to $t = 1$). The chain rule then shows that

$$\frac{d}{dt}f(z(t)) = f'(z(t))\frac{dz(t)}{dt} = (z_1 - z_0)f'(z(t)) = 0$$

since f' is zero in D. It follows that both the real and imaginary parts of $f(z(t))$ have zero t derivative and are therefore constant on $\overbrace{P_0\ P_1}$. The same argument applied to the other line segments of the polygon allows us to conclude that $f(\zeta) = f(z_0)$ and that f is constant on D.

Theorem 3.4 *Suppose that $f : D \to \mathbf{C}$ is analytic and that $|f(z)|$ is constant on D. Then f is a constant function on D.*

Put $f(z) = u + iv$ so that $u^2 + v^2 = C$, where C is a constant. Differentiation gives

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0, \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0\ \forall z \in D.$$

It follows from the Cauchy-Riemann equations that

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0, \quad v\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0\ \forall z \in D.$$

If u and v are both zero at some point then $C = 0$ and f is everywhere zero. Otherwise, we conclude that the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are both zero in D; also $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are both zero in D. It follows that $f'(z) = 0$ and that f is constant on D.

3.5 Analytic functions and Laplace's equation

Suppose that $f : D \to \mathbf{C}$ is analytic and that $f(z) = u(x, y) + iv(x, y)$, in the usual notation.

It can be proved, by complex integration methods, that f is in fact differentiable to all orders in D. It follows that u and v have continuous partial derivatives to all orders and the commutative law of partial differentiation holds. On this basis we have, starting from the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x}$$

It follows immediately that u satisfies the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

It is clear, therefore, that the real and imaginary parts of an analytic function satisfy Laplace's equation; they are *harmonic* functions.

Continuing in the same notation we note

Theorem 3.5 *The curves $u(x, y) = C_1$, $v(x, y) = C_2$ (where C_1, C_2 are parameters) define an orthogonal network of curves.*

To see this, differentiate the equation $u(x, y) = C_1$ with respect to x (bearing in mind that in this case y is an implicit function of x) to obtain

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0.$$

Similarly, the equation $v(x, y) = C_2$ gives

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0.$$

These equations show that the products of the gradients of the two curves at the point of intersection is equal to

$$\left(\frac{-u_x}{u_y}\right)\left(\frac{-v_x}{v_y}\right) = \left(\frac{-v_y}{v_x}\right)\left(\frac{v_x}{v_y}\right) = -1,$$

as required.

Note that for our calculation to make sense the first order partial derivatives of u and v with respect to x, y shouldn't be zero. This is the very condition laid down in the implicit function theorem for $u(x, y) = C_1$ to define y as an implicit function of x. (similarly for $v(x, y) = C_2$)

3.6 Analytic functions and vector fields

Finally, we note that associated with an analytic function f is a two-dimensional irrotational solenoidal vector field $\mathbf{q}$. To make this clear we write $f(z) = \phi(x, y) + i\psi(x, y)$, $z = x + iy$. (we make the change of notation to link up with the discussion at the start of this chapter). Put $\mathbf{q}(x, y) = -\nabla\phi$. Then curl $\mathbf{q} = \mathbf{0}$ and div$\mathbf{q} = 0$ since ϕ, being the real part of an analytic function, satisfies Laplace's equation. It is easy to check from the Cauchy-Riemann equations that ψ is the associated stream function.

Chapter 4

The elementary functions

4.1 The exponential function

The exponential function $\exp(z)$ is defined by

$$\exp(z) = \mathrm{e}^x(\cos y + i\sin y),\ z = x + iy \in \mathbf{C}\,.$$

Note that on the real axis exp reduces to the familiar real exponential function: $\exp(x) = \mathrm{e}^x\ \ \forall x \in \mathbf{R}\,.$

It follows easily from this definition that

$$\exp(z_1)\exp(z_2) = \exp(z_1 + z_2),\ \forall z_1, z_2 \in \mathbf{C}\,.$$

To check this put $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$. Then

$$\begin{aligned}\exp(z_1)\exp(z_2) &= \mathrm{e}^{x_1}(\cos y_1 + i\sin y_1)\,\mathrm{e}^{x_2}(\cos y_2 + i\sin y_2)\\ &= \mathrm{e}^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2)\end{aligned}$$

Now

$$(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) = \cos(y_1 + y_2) + i\sin(y_1 + y_2)$$

as follows by multiplying out the brackets and using standard formulae from trigonometry. Consequently

$$\exp(z_1)\exp(z_2) = \mathrm{e}^{x_1+x_2}(\cos(y_1 + y_2) + i\sin(y_1 + y_2)) = \exp(z_1 + z_2)$$

since $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

It is clear that $\exp(z)\exp(-z) = \exp(0) = 1$ so that $\exp(-z) = 1/\exp(z)$.

On the basis of the properties of exp we are justified in writing $\exp(z) \equiv \mathrm{e}^z$, treating z in e^z according to the rules of indices. In this notation we have

$$\mathrm{e}^z = \mathrm{e}^x(\cos y + i\sin y),\ z = x + iy \in \mathbf{C}\,.$$

Notice that when θ is real $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$ so that our notation is consistent with the notation introduced in Chapter 2.

Observe that when k is any integer $\mathrm{e}^{z+2k\pi i} = \mathrm{e}^{z}\,\mathrm{e}^{2k\pi i} = \mathrm{e}^{z}$ so that the exponential is periodic with (fundamental) period equal to $2\pi i$.

The exponential function has no zeros. This is clear since $|\,\mathrm{e}^{iy}\,| = \sqrt{\cos^2 y + \sin^2 y} = 1,\ \forall y \in \mathbf{R}$ and therefore $|\,\mathrm{e}^{z}\,| = |\,\mathrm{e}^{x}\,\mathrm{e}^{iy}\,| = \mathrm{e}^{x}$; of course, e^{x} is never zero for $x \in \mathbf{R}$. Confirmation that our definition of the exponential is a good one is obtained by verifying that the exponential is analytic in $\mathbf{C}$; it is an entire function. To check this put

$$\mathrm{e}^{z} = u + iv, \quad u = \mathrm{e}^{x}\cos y,\ v = \mathrm{e}^{x}\sin y.$$

Then

$$\frac{\partial u}{\partial x} = \mathrm{e}^{x}\cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\,\mathrm{e}^{x}\sin y = -\frac{\partial v}{\partial x}.$$

u, v therefore satisfy the Cauchy-Riemann equations. Moreover, the first order partial derivatives of u, v with respect to the variables x, y are continuous. It follows that the exponential is an analytic function by theorem 3.2. Its derivative is given by

$$\frac{d}{dz}\,\mathrm{e}^{z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \mathrm{e}^{x}\cos y + i\,\mathrm{e}^{x}\sin y = \mathrm{e}^{x}(\cos y + i\sin y) = \mathrm{e}^{z}\,.$$

We have therefore established the fundamental property that the exponential is analytic in $\mathbf{C}$ and satisfies $\frac{d}{dz}\,\mathrm{e}^{z} = \mathrm{e}^{z}$.

4.2 Trigonometric and hyperbolic functions

Our definitions are motivated by formulae 2.1 of Chapter 2. For $z \in \mathbf{C}$ we define $\sin z$ and $\cos z$ by

$$\sin z = \frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz}), \quad \cos z = \frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz}) \tag{4.1}$$

The definition of the hyperbolic functions is obvious:

$$\sinh z = \frac{1}{2}(\mathrm{e}^{z} - \mathrm{e}^{-z}), \quad \cosh z = \frac{1}{2}(\mathrm{e}^{z} + \mathrm{e}^{-z}) \tag{4.2}$$

It follows at once that sin, cos, sinh, cosh are analytic in $\mathbf{C}$ and that they obey the usual differentiation rules familiar from elementary calculus. For example, $\frac{d}{dz}\sin z = \frac{1}{2i}(i\,\mathrm{e}^{iz} - (-i)\,\mathrm{e}^{-iz}) = \cos z$, and so on.

As in trigonometry, we can define $\tan z$ by $\tan z = \sin z/\cos z$ (we omit those points z where $\cos z = 0$); it follows in the usual way that tan is analytic, except for z such that $\cos z = 0$, and that $\frac{d}{dz}\tan z = \sec^2 z$, $\sec z$ being defined in terms of $\cos z$ as in trigonometry. It should be clear how to define functions such as $\operatorname{sech} z$, $\coth z$ etc.

The familiar identities for the trigonometric and hyperbolic functions also hold in the complex case, for example $\cos^2 z + \sin^2 z = 1$. Note however, that although $|\cos x| \leq 1,\ \forall x \in \mathbf{R}$ it is *not* true that $|\cos z| \leq 1$, for all $z \in \mathbf{C}$. In fact, $|\cos z|$ can be very large indeed!

4.3 The logarithmic function

We are familiar with the real logarithm $\ln : \mathbf{R}^+ \to \mathbf{R}$. It is the inverse of the real exponential function; if $x = \mathrm{e}^t,\ t \in \mathbf{R}$ then $\ln x = t$. Can we adopt a similar approach to define the complex logarithm? Given the equation $\mathrm{e}^w = z$ we should like to write $w = \log z$, where log denotes a complex logarithm – but this poses problems as we now show.

We can regard the equation $z = \mathrm{e}^w$ as defining a map of the complex w-plane into the complex z-plane. Put $w = \alpha + i\beta$ and $z = x + iy$ so that $\mathrm{e}^\alpha(\cos\beta + i\sin\beta) = x + iy$. The image of the infinite strip $A_1 = \{\alpha \in \mathbf{R},\ 0 \leq \beta < 2\pi\}$ is the whole complex z-plane, the point $z = 0$ being omitted. We can see this by noting that the image of the section $\{\alpha \text{ fixed},\ 0 \leq \beta < 2\pi\}$ is in fact a circle of radius e^α, since $|z| = \mathrm{e}^\alpha$. As α takes on all real values we obtain a family of circles which cover the whole z-plane, with the exception of $z = 0$. (this reflects the fact that e^α is never zero). The same argument shows that e^w maps the infinite strips A_2, A_3 ... where

$$A_2 = \{\alpha \in \mathbf{R},\ 2\pi \leq \beta < 4\pi\},\ A_3 = \{\alpha \in \mathbf{R},\ 4\pi \leq \beta < 6\pi\},\ldots$$

onto the whole complex z-plane, the point $z = 0$ being omitted. The question then arises: Given z, what value of w do we assign? Do we choose the relevant value of $w \in A_1$, the relevant value of $w \in A_2,\ldots$? It is clear that if we attempt to parallel the procedure adopted with real variables we shall end up with a muti-valued complex logarithm.

With these thoughts in mind we make the following definition.

Definition 4.1 *For $z \in \mathbf{C} - \{0\}$ a logarithm of z is any particular solution of the equation $\mathrm{e}^w = z$.*

We can compute the relevant solutions as follows.

Continuing in the same notation, with $w = \alpha + i\beta$ and $z = x + iy = r(\cos\theta + i\sin\theta) = r\,\mathrm{e}^{i\theta}$ we have

$$\mathrm{e}^{\alpha+i\beta} = r\,\mathrm{e}^{i\theta},\ (r = |z|)$$

This gives

$$\mathrm{e}^\alpha = r,\ \mathrm{e}^{i(\beta-\theta)} = 1,\ \Rightarrow \alpha = \ln r = \ln|z|,\ \beta - \theta = 2k\pi,\ \text{where } k \text{ is an integer.}$$

Note that these formulae only make sense provided we exclude the point $z = 0$. We therefore have

$$w = \ln|z| + i(\theta + 2k\pi) \tag{4.3}$$

Equivalently

$$w = \log z = \ln|z| + i\arg z \tag{4.4}$$

The multi-valued character of $\log z$ is due to the multi-valued nature of $\arg z$. According to our definition, a logarithm of z is any one of the infinite number of values permitted by equation 4.3. Each of the permitted values constitutes a *branch* of the logarithm. However, in analysis functions are required to be single valued. For this reason one usually chooses a particular branch of the logarithm by restricting oneself to a particular choice of $\arg z$ so that multi-valuedness does not occur. If one moves round the origin, allowing the argument of z to increase by 2π, one moves from one branch of the logarithm to another - single-valuedness has been lost. For this reason $z = 0$ is often referred to as a *branch point* of the logarithm. In order to ensure that we keep to one branch when dealing with $\log z$ it is therefore common practice to introduce a suitable *cut* in the complex plane, in the form of a straight line extending from $z = 0$ to infinity, across which we are not permitted to pass.

These considerations may be important in the area of complex integration where we often integrate round a closed curve; if such a curve encloses the point $z = 0$ then the argument of z will increase by 2π if we move round the curve and return to the starting point.

One branch of the logarithm which is frequently used is the so-called *principal value*, often denoted by Log z, which is defined by the requirement that the argument of z has its principal value Arg z. Thus

$$\text{Log } z = \ln|z| + i\,\text{Arg } z, \ -\pi < \text{Arg } z \leq \pi.$$

Note that Log z reduces to the familiar real logarithm $\ln x$ on the real-axis. It should be clear that great care is required when working with the complex logarithm. We note, for example, the following properties:

- $\mathrm{e}^{\log z} = z$
- $\log \mathrm{e}^z = z + 2k\pi i, \ k$ an integer
- $\log z_1 z_2 = \log z_1 + \log z_2 + 2k\pi i, \ k$ an integer.
- $\log(1/z) = -\log z + 2k\pi i, \ k$ an integer.

The first property is true by construction.

To check the second property write $\mathrm{e}^z = \mathrm{e}^x(\cos y + i\sin y)$ so that

$\log \mathrm{e}^z = \ln|\,\mathrm{e}^z\,| + i \arg \mathrm{e}^z = \ln \mathrm{e}^x + i(y + 2k\pi), k$ an integer. This gives
$\log \mathrm{e}^z = (x + iy) + 2k\pi i = z + 2k\pi i.$

The third property follows in a similar way:

$$\log z_1 z_2 \quad = \quad \ln|z_1 z_2| + i \arg z_1 z_2 = \ln|z_1| + \ln|z_2| + i \arg z_1 z_2$$

Hence

$$\log z_1 z_2 \quad = \quad \ln|z_1| + \ln|z_2| + i \arg z_1 + i \arg z_2 + 2k\pi i = \log z_1 + \log z_2 + 2k\pi i,$$

where $k = 0, \pm 1, \pm 2, \cdots$

Checking the fourth property is left as an exercise for the reader.

As an illustration of the definition, consider the following.

$\text{Log}\ i(-1 + i) = \text{Log}\,(-1 - i) = \ln\sqrt{2} - i(3\pi)/4.$

However, $\text{Log}\ i = i(\pi/2)$ and $\text{Log}\,(-1 + i) = \ln\sqrt{2} + i(3\pi/4)$. We see that
$\text{Log}\ i(-1 + i) \neq \text{Log}\ i + \text{Log}\,(-1 + i)$.

Note that the discrepancy, the difference between $\text{Log}\ i + \text{Log}\,(-1+i)$ and $\text{Log}\ i(-1+i)$, is equal to $2\pi i$, a multiple of $2\pi i$, as it should be. We see that the the familiar rules of logarithms do not apply in this case, a fact which merely emphasises the need for great care in dealing with complex logarithms. If in doubt, go back to the definition!

Now suppose that D is a domain in the cut plane (see above), so that no point of the cut belongs to D; in particular $0 \notin D$. We now show that a logarithm is analytic in D and that $\frac{d}{dz} \log z = \frac{1}{z}$. (If we were considering the principal value of the logarithm we could introduce a cut along the negative real axis from $-\infty$ to 0) The following argument isn't a model of rigour, however!

Write

$$\log z = \frac{1}{2}\ln(x^2 + y^2) + i \arg z = u + iv, \quad u = \frac{1}{2}\ln(x^2 + y^2), \quad \tan v = \frac{y}{x}.$$

Differentiation gives

$$\sec^2 v \frac{\partial v}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial v}{\partial x} = -\frac{y}{x^2}\frac{1}{1 + y^2/x^2} = -\frac{y}{x^2 + y^2},$$

$$\sec^2 v \frac{\partial v}{\partial y} = \frac{1}{x}, \quad \frac{\partial v}{\partial y} = \frac{1}{x}\frac{x^2}{x^2 + y^2} = \frac{x}{x^2 + y^2}.$$

Similarly

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}.$$

The first order partial derivatives of u, v with respect to x and y are continuous in D ($z = 0$ is not in D) and the Cauchy-Riemann equations are satisfied. It follows from theorem 3.2 that $\log z$ is analytic in D and that

$$\frac{d}{dz}\log z = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x - iy}{x^2+y^2} = \frac{1}{z}, \quad z \in D.$$

4.4 Complex powers

For $z, \zeta \in \mathbf{C}$ ($z \neq 0$) we define z^ζ by $z^\zeta = \mathrm{e}^{\zeta \log z}$.

This is a natural definition which parallels the definition of x^α for $x > 0$ and $\alpha \in \mathbf{R}$.

The value of z^ζ will depend on the choice of $\log z$; there are an infinite number of possibilities, of course. The *principal value* of z^ζ is obtained by assigning the principal value of $\log z$ i.e. $\mathrm{Log}\ z$. z^ζ will be analytic in any domain which doesn't include the point $z = 0$. We shall not discuss complex powers further, beyond mentioning that great care is needed when dealing with them. One cannot *assume* that the ordinary rules of indices apply (they don't!) and if in doubt the guiding principle should be — go back to the definition and work from there. Careless assumptions can quickly lead to false conclusions. For example

$$\mathrm{e}^{2\pi i} = \mathrm{e}^{4\pi i} = \mathrm{e}^{6\pi i} = \mathrm{e}^{8\pi i} = \cdots$$

since each term is equal to 1. If we carelessly *assumed* that we could raise each to the power i we might conclude that

$$\mathrm{e}^{-2\pi} = \mathrm{e}^{-4\pi} = \mathrm{e}^{-6\pi} = \cdots$$

It would follow that $2 = 4 = 6 = \cdots$ — that all integers are equal!

As an illustration, the principle value of i^i is $\mathrm{e}^{i\,\mathrm{Log}\ i} = \mathrm{e}^{i(0+i\pi/2)} = \mathrm{e}^{-\pi/2}$.

As a final illustration of the potential hazards we note that

$$(z_1 z_2)^\zeta = \mathrm{e}^{\zeta \log(z_1 z_2)} = \mathrm{e}^{\zeta(\log z_1 + \log z_2 + 2k\pi i)}, \ k \text{ an integer}$$

so that

$$(z_1 z_2)^\zeta = z_1^\zeta z_2^\zeta\, \mathrm{e}^{\zeta(2k\pi i)}.$$

Generally, the right-hand side isn't equal to $z_1^\zeta z_2^\zeta$.

Chapter 5

Complex integration

We start with some definitions.

Definition 5.1 *A curve* γ *is a continuous function* $\gamma : [a,b] \to \mathbf{C}$ *for some* $a \leq b$.

Definition 5.2 *The set of points* $\{z \in \mathbf{C} : z = \gamma(t),\ a \leq t \leq b\}$ *is called the trace of* γ*, written* tr γ.

These are the definitions favoured by pure mathematicians and there are in fact good reasons for making a distinction between a curve and its trace, as the following simple examples show. Nevertheless, especially when we come to consider complex integration, we shall frequently refer to 'a curve' when, strictly speaking, we mean its trace.

Example 5.1 *Let* $\gamma_1(t) = \mathrm{e}^{2\pi i t},\ 0 \leq t \leq 1$.

In this case tr $\gamma_1 = \{z : |z| = 1\}$ — which is just the unit circle.

Example 5.2 *Consider now* $\gamma_2(t) = \mathrm{e}^{-4\pi i t},\ 0 \leq t \leq 1$.

Then tr $\gamma_2 = \{z : |z| = 1\}$ — also the unit circle. However, γ_1 and γ_2 are different curves; γ_1 goes tround the unit circle once in the anti-clockwise sense whereas γ_2 goes twice round the unit circle in the clockwise sense!

Definition 5.3 *Let* $\gamma : [a,b] \to \mathbf{C}$ *be a curve. We say that* γ *is closed if* $\gamma(a) = \gamma(b)$. γ *is said to be simple if it does not cross itself i.e* $\not\exists t', t'' \in (a,b)$ $(t' \neq t'')$ *such that* $\gamma(t') = \gamma(t'')$.

Definition 5.4 *A simple closed curve is one which is simple and closed.*

We note, for example, that if $\gamma(t) = \mathrm{e}^{-5\pi i t},\ 0 \leq t \leq 1$ then tr γ is the unit circle but γ is neither closed nor simple. (γ goes twice round the unit circle and then half way round again, in a clockwise sense)

Definition 5.5 *Let* $\gamma : [a,b] \to \mathbf{C}$ *be a curve. The reverse curve* $\widetilde{\gamma}$ *is defined by* $\widetilde{\gamma} : [a,b] \to \mathbf{C}$*, where* $\widetilde{\gamma}(t) = \gamma(a+b-t)$.

Definition 5.6 *A curve γ is smooth if $\gamma'(t)$ exists and is continuous (at a and b we mean right and left derivatives respectively).*

If $\gamma(t) = x(t) + iy(t)$ then γ is smooth if and only if x' and y' exist and are continous.

Definition 5.7 *A contour is a piecewise smooth curve.*

Thus, if $\gamma : [a, b] \to \mathbf{C}$ is a contour we can partition $[a, b]$ by points $a_0, a_1, \dots, a_{m-1}, a_m$ (for some m) such that $a = a_0 < a_1 < a_2 < \dots < a_{m-1} < a_m = b$ and for which $\gamma : [a_i, a_{i+1}] \to \mathbf{C}$ is smooth for $0 \le i \le m-1$. We denote the restriction of γ to $[a_i, a_{i+1}]$ by γ_i and write in an obvious notation $\gamma = \gamma_0 + \gamma_1 + \cdots + \gamma_{m-1}$.

5.1 Length of a curve

Suppose that $\gamma : [a, b] \to \mathbf{C}$ is smooth. We partition $[a, b]$ by points $t_0, t_1, \dots t_{n-1}, t_n$ such that $a = t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n = b$. The points $z_j = \gamma(t_j)$ define a polygon with vertices at $z_0, z_1, \dots, z_n$. This polygon has length

$$\sum_{j=1}^{n} |z_j - z_{j-1}| = \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| = \sum_{j=1}^{n} \left| \frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}} \right| (t_j - t_{j-1}).$$

The right hand-side is like a Riemann sum and

$$\left| \frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}} \right| \sim |\gamma'(t_j)|, \text{ where } |\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

We might expect that as we refine the partition of $[a, b]$, introducing further points of dissection that, as $\max(t_j - t_{j-1}) \to 0$, our expression for the polygonal length will tend to $\int_a^b |\gamma'(t)|\, dt$, and this can indeed be proved rigorously. On this basis we make the following definition:

Definition 5.8 *The length of a smooth curve $\gamma : [a, b] \to \mathbf{C}$ is $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$.*

If γ is a contour, $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_s$, with the γ_j smooth, then we define
$L(\gamma) = L(\gamma_1) + L(\gamma_2) + \cdots + L(\gamma_s)$.

Example 5.3 *Let $\gamma(t) = \mathrm{e}^{2\pi i t}$, $0 \le t \le 1$.*

Then

$$L(\gamma) = \int_0^1 |\gamma'(t)|\, dt = \int_0^1 |2\pi i\, \mathrm{e}^{2\pi i t}|\, dt = 2\pi \int_0^1 dt = 2\pi,$$

a result which should cause no surprise — it is just the length of the unit circle.

Example 5.4 *Suppose that*

$$\gamma(t) = \begin{cases} \mathrm{e}^{2\pi i t}, & 0 \le t \le 1 \\ 1, & 1 \le t \le 2 \\ \mathrm{e}^{-4\pi i t}, & 2 \le t \le 3 \end{cases}$$

Then γ is a contour (check this) and

$$\gamma'(t) = \begin{cases} 2\pi i\, \mathrm{e}^{2\pi i t}, & 0 \le t < 1 \\ 0, & 1 < t < 2 \\ -4\pi i\, \mathrm{e}^{-4\pi i t}, & 2 < t \le 3 \end{cases}$$

At $t = 1$ the left derivative of γ is $2\pi i$ and the right derivative 0. At $t = 2$ the left derivative is 0 and the right derivative $-4\pi i$.

$$L(\gamma) = \int_0^1 |\gamma'(t)|\, dt + \int_1^2 |\gamma'(t)|\, dt + \int_2^3 |\gamma'(t)|\, dt = \int_0^1 2\pi\, dt + \int_1^2 0\, dt + \int_2^3 4\pi\, dt = 6\pi.$$

This is a reflection of the fact that γ starts at $z = 1$, goes round the unit circle once in the anti-clockwise sense, $(0 \le t \le 1)$, stops at the point $z = 1$, $(1 \le t \le 2)$, and then goes twice round the unit circle in the clockwise sense, $(2 \le t \le 3)$, returning to the starting point $z = 1$.

Before considering integration along a contour we note that if $f : [a, b] \to \mathbf{C}$ is a continuous complex valued function such that $f(t) = f_1(t) + i f_2(t)$, where f_1, f_2 are real valued on $[a, b]$, then

$$\int_a^b f(t)\, dt = \int_a^b f_1(t)\, dt + i \int_a^b f_2(t)\, dt.$$

This can be proved in a straightforward manner starting from the definition of the integral in terms of Riemann sums and is a reflection of the fact that integration is a linear operation.

5.2 Integration along a contour

Let $\gamma : [a, b] \to \mathbf{C}$ be a smooth curve and suppose that f is a function which is continuous in a region containing $\operatorname{tr} \gamma$. We wish to define the integral of f along the curve γ, $\int_\gamma f(z)\, dz$.

A natural way to proceed is to partition $[a, b]$ as above by points $t_0, t_1, \ldots t_{n-1}, t_n$ such that $a = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n = b$. The points $z_j = \gamma(t_j)$ define a polygon with vertices at $z_0, z_1, \ldots, z_n$. We may pick in each subinterval $[a_{j-1}, a_j]$ an arbitrary point ξ_j and form the sum

$$\sum_{j=1}^n f(\zeta_j = \gamma(\xi_j))(z_j - z_{j-1}) = \sum_{j=1}^n f(\zeta_j = \gamma(\xi_j)) \frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}} (t_j - t_{j-1}).$$

It seems highly plausible, and can be proved, that as we take dissection limits of these sums, as the length of the longest interval tends to zero, they tend to $\int_a^b f(\gamma(t))\gamma'(t)\, dt$. We therefore take this integral as our *definition* of $\int_\gamma f(z)\, dz$. To be precise:

Definition 5.9 *Let $\gamma : [a, b] \to \mathbf{C}$ be a smooth curve and suppose that f is a function which is continuous in a region containing* $\operatorname{tr} \gamma$. *Then*

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Since $z = \gamma(t)$ it may perhaps appeal to our intuition if we write the integral in the equivalent notation

$$\int_a^b f(z(t))\frac{dz}{dt}\,dt, \quad z(t) = \gamma(t).$$

If, in the usual notation, we write $f(z) = u(x,y) + iv(x,y)$, $z(t) = \gamma(t)$ we have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b \Big\{u(x(t),y(t)) + iv(x(t),y(t))\Big\}(x'(t) + iy'(t))\,dt \\ &= \int_a^b \Big\{u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)\Big\}\,dt \\ &+ i\int_a^b \Big\{v(x(t),y(t))x'(t) + u(x(t),y(t))y'(t)\Big\}\,dt.\end{aligned}$$

If γ is a contour we make the definition

Definition 5.10

$$\int_\gamma f(z)\,dz = \sum_{j=1}^n \int_{\gamma_j} f(z)\,dz,$$

where the γ_j are the smooth parts of γ.

The following is a very important example whose significance will become clear later on.

Example 5.5 *Let* $\gamma(t) = r\,\mathrm{e}^{it},\ 0 \le t \le 2\pi,\ r > 0,\quad f(z) = 1/z$. *Then* $\int_\gamma dz/z = 2\pi i$.

We have

$$\int_\gamma f(z)\,dz = \int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{1}{r\,\mathrm{e}^{it}}\frac{d}{dt}(r\,\mathrm{e}^{it})\,dt = \int_0^{2\pi} \frac{1}{r\,\mathrm{e}^{it}}(ri\,\mathrm{e}^{it})\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

In this example we've integrated $1/z$ round the circle centre 0 and radius r in the positive (i.e. anti-clockwise) sense; the integral is equal to $2\pi i$, irrespective of the value of $r > 0$.

We note the following.

Theorem 5.1 *Let γ be a contour and suppose that f, g are continuous in a region which contains* $\mathrm{tr}\,\gamma$ *and that $\alpha, \beta \in \mathbf{C}$ are constants. Then*

- $$\int_\gamma (\alpha f(z) + \beta g(z))\,dz = \alpha\int_\gamma f(z)\,dz + \beta\int_\gamma g(z)\,dz,$$
- $$\int_{\widetilde{\gamma}} f(z)\,dz = -\int_\gamma f(z)\,dz,$$

  where $\widetilde{\gamma}$ is the reverse curve to γ

The proof of the first result follows from the definition and reflects the linearity of the operation of integration. As regards the second result, it is enough to suppose that γ is smooth, with $\gamma : [a,b] \to \mathbf{C}$ in the now familiar notation. We have

$$\int_{\widetilde{\gamma}} f(z)\,dz = \int_a^b f(\widetilde{\gamma(t)})\frac{d\widetilde{\gamma(t)}}{dt}\,dt = \int_a^b f(\gamma(a+b-t))\frac{d\gamma(a+b-t)}{dt}\,dt.$$

Putting $s = a+b-t,\ dt = -ds$ gives

$$\int_{\widetilde{\gamma}} f(z)\,dz = \int_b^a f(\gamma(s))\frac{d\gamma(s)}{ds}\,ds = -\int_a^b f(\gamma(s))\frac{d\gamma(s)}{ds}\,ds = -\int_\gamma f(z)\,dz.$$

This means that if we integrate along a curve in the opposite sense we change the sign of the integral.

Note: It is not hard to verify that $\int_\gamma f$ is independent of the choice of parametrisation of γ. In essence it depends on applying the chain rule.

Example 5.6 *Evaluate* $\int_\gamma \overline{z}\,dz$, *where* $\gamma = \gamma_1 + \gamma_2$, γ_1 *being the line segment from* 1 *to* 0 *and* γ_2 *being the line segment from* 0 *to* $2+2i$.

On $\gamma_1,\ \gamma_1(t) = 1-t,\ 0 \le t \le 1$ and

$$\int_{\gamma_1} \overline{z}\,dz = \int_0^1 (1-t)(-1)\,dt = -\frac{1}{2}.$$

[More formally, $\int_{\gamma_1} \overline{z}\,dz = \int_{\gamma_1}(x-iy)(dx+idy) = \int_1^0 x\,dx = -\frac{1}{2}$, since $y=0$ on γ_1.]

On $\gamma_2,\ \gamma_2(t) = 0 + t(2+2i) = 2(1+i)t,\ 0 \le t \le 1$ and $\int_{\gamma_2} \overline{z}\,dz = \int_0^1 2(1-i)t(2+2i)\,dt$ $= 4 \times 2\int_0^1 t\,dt = 4$ so that $\int_\gamma \overline{z}\,dz = \int_{\gamma_1}\overline{z}\,dz + \int_{\gamma_2}\overline{z}\,dz = 4 - \frac{1}{2} = \frac{7}{2}$.

[More formally, note that we can use x to parametrise γ_2 since $y = x$ on this straight-line segment, $0 \le x \le 2$. We then have

$$\int_{\gamma_2} \overline{z}\,dz = \int_{\gamma_2}(x-iy)(dx+idy) = \int_0^2 x(1-i)(dx+idx) = (1-i)(1+i)\int_0^2 x\,dx = 4,$$

as found above.]

We know that for continuous real valued functions $f : [a,b] \to \mathbf{R}$ that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

We now extend this to complex integrals.

Theorem 5.2 *Let γ be a contour and suppose that f is a complex function which is continuous in a region which contains* tr γ. *Then*

$$\left|\int_\gamma f(z)\,dz\right| \le \int_a^b |f(z(t))|\left|\frac{dz}{dt}\right| dt,$$

where $\gamma : [a,b] \to \mathbf{C}$ and $z(t) = \gamma(t)$.

Moreover, if $|f(\zeta)| \le M,\ \forall\zeta \in \mathrm{tr}\ \gamma$, then $|\int_\gamma f(z)\,dz| \le ML(\gamma)$.

To prove this result we first note that if $\int_\gamma f = 0$ then there is nothing to prove. We therefore suppose that $\int_\gamma f \ne 0$ and let $\theta = \arg \int_\gamma f$. We then have $\int_\gamma f = \mathrm{e}^{i\theta}|\int_\gamma f|$, so that

$$\begin{aligned}\left|\int_\gamma f\right| &= \mathrm{e}^{-i\theta}\int_\gamma f = \int_a^b \mathrm{e}^{-i\theta}\, f(\gamma(t))\gamma'(t)\,dt\\ &= \int_a^b \Re(\mathrm{e}^{-i\theta}\, f(\gamma(t))\gamma'(t))\,dt + i\int_a^b \Im(\mathrm{e}^{-i\theta}\, f(\gamma(t))\gamma'(t))\,dt\end{aligned}$$

The integral on the left is a real number, as is the first integral on the right (being the integral of a real function), so it follows that the second integral on the right must be zero. We then have

$$\left|\int_\gamma f\right| = \int_a^b \Re(\mathrm{e}^{-i\theta}\, f(\gamma(t))\gamma'(t))\,dt \le \int_a^b |(\mathrm{e}^{-i\theta}\, f(\gamma(t))\gamma'(t))|\,dt$$

since $|\Re\zeta| \le |\zeta|$ for any complex number ζ. We therefore obtain

$$\left|\int_\gamma f(z)\,dz\right| \le \int_a^b |f(z(t))|\left|\frac{dz}{dt}\right| dt,$$

as required. The last part follows at once since if $|f| \le M|$ on tr γ then

$$\left|\int_\gamma f(z)\,dz\right| \le \int_a^b M|\gamma'(t)|\,dt = M\int_a^b |\gamma'(t)|\,dt = ML(\gamma).$$

The following is an important theorem.

Theorem 5.3 *Suppose that f' exists and is continuous on* tr γ, *where (in the usual notation) $\gamma : [a,b] \to \mathbf{C}$ is a contour. Then*

$$\int_\gamma f'(z)\,dz = f(z = \gamma(b)) - f(z = \gamma(a)).$$

To prove this we asume that γ is smooth (in the case of a contour we just add up the contributions from each of the smooth bits) and write $f(z) = u(x,y) + iv(x,y)$,

$z = \gamma(t) = x(t) + iy(t)$. Then, by definition,

$$\begin{aligned}
\int_\gamma f'(z)\,dz &= \int_a^b f'(\gamma(t))\gamma'(t)\,dt = \int_a^b \frac{d}{dt} f(\gamma(t))\,dt \\
&= \int_a^b \left(\frac{d}{dt} u(x(t), y(t)) + i \frac{d}{dt} v(x(t), y(t)) \right) dt \\
&= \Big[u(x(t), y(t)) + iv(x(t), y(t)) \Big]_{t=a}^{t=b}
\end{aligned}$$

by the fundamental theorem of the integral calculus. The stated result has been established.

This result motivates the following definition

Definition 5.11 *Suppose that D is a domain and that $f : D \to \mathbf{C}$ is continuous. A map $F : D \to \mathbf{C}$ such that $F' = f$ on D is called a primitive for f on D.*

We now have as a corollary of our theorem:

If F is a primitive for f on a domain D and γ is any contour whose trace lies in D then

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is *closed* and f has a primitive then $\int_\gamma f(z)\,dz = 0$.

This theorem is clearly important since it enables us to compute many integrals in a manner which is familiar from elementary calculus. As an example, let $f(z) = z^n$, where n is a positive integer. If γ is any smooth curve connecting the points z_1 and z_2 then

$$\int_\gamma f(z)\,dz = \frac{1}{n+1}[z_2^{n+1} - z_1^{n+1}]$$

since $F(z) = \frac{1}{n+1} z^{n+1}$ is a primitive for z^n.

Example 5.7 *Evaluate $\int_\gamma \frac{dz}{z}$, where γ is the straight line from $z = 2$ to $z = 1 + i\sqrt{3}$.*

Log z is a primitive for $\frac{1}{z}$ in D (see diagram).

We then have

$$\int_\gamma \frac{dz}{z} = \Big[\text{Log } z\Big]_2^{1+i\sqrt{3}} = \text{Log}\,(1+i\sqrt{3}) - \text{Log } 2$$

$$= \ln\sqrt{(4)} + i\,\text{Arg}\,(1+i\sqrt{3}) - (\ln 2 + i\,\text{Arg } 2) = \ln 2 + i\frac{\pi}{3} - \ln 2 = i\frac{\pi}{3}.$$

We also have the following result (change of variable)

Theorem 5.4 *Suppose that* $\gamma_1 : [a,b] \to \mathbf{C}$ *is a smooth curve and that* $\gamma_2(t) = \zeta(\gamma_1(t)),\ t \in [a,b]$*, where* ζ *is a smooth complex valued function in a region which contains the trace of* γ_1*. Let* f *be a continuous complex valued function in a region which contains the trace of* γ_2*. Then*

$$\int_{\gamma_2} f(\zeta)\,d\zeta = \int_{\gamma_1} f(\zeta(z))\frac{d\zeta}{dz}\,dz$$

For, we can write

$$\int_{\gamma_2} f(\zeta)\,d\zeta = \int_a^b f(\zeta(\gamma_1(t)))\frac{d\zeta}{dt}\,dt = \int_a^b f(\zeta(\gamma_1(t)))\frac{d\zeta}{dz}\frac{dz}{dt}\,dt \quad (z = \gamma_1(t))$$

$$= \int_{\gamma_1} f(\zeta(z))\frac{d\zeta}{dz}\,dz$$

where we've used the chain rule of differentiation and also our definition of $\int_{\gamma_k}$ $(k = 1,2)$.

Chapter 6

Cauchy's theorem

6.1 Cauchy's theorem

Theorem 6.1 *(Cauchy's theorem) Suppose that D is a domain, that $f : D \to \mathbf{C}$ is analytic, and that γ is a simple smooth closed curve such that* tr γ *and its interior lie in the domain D. Then $\int_\gamma f(z)\,dz = 0$.*

There are many versions of this theorem which make different assumptions about the curve γ; for example the theorem can be proved on the assumption that the curve γ is rectifiable (roughly speaking this means that it has a finite length); this does not require the fairly strong assumption that the curve be smooth. Rigorous proofs, of even the simplest version of the theorem, are not easy.

Cauchy's theorem is the fundamental theorem of complex variable theory on which the whole structure depends. A proof of the theorem which requires a stronger set of assumptions than is actually necessary depends on Stokes'[1] theorem for the plane.

Following on in the above notation we state

Theorem 6.2 *(Stokes' theorem for the plane) Suppose that $P, Q : D \to \mathbf{R}$ are functions with continuous first order partial derivatives ($\mathbf{C}^1$ functions). Then*

$$\int_{\mathcal{A}} \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right) dx\,dy = \int_\gamma (Q\,dx + P\,dy)$$

where $\mathcal{A}$ is the area enclosed by tr γ *and the line integrals are takem in the positive (anti-clockwise) sense.*

Before proceeding, we mention that the line integrals are to be interpreted along the lines previously introduced for complex integrals. Thus, we assume that γ is parametrised in

[1]Sir George Stokes (1819-1903) Professor of Mathematics at Cambridge from 1849

terms of a real variable t. $\int_\gamma Q(x,y)dx$ is then to be understood as $\int Q((x(t),y(t))\frac{dx}{dt}\,dt$ integrated over t between the appopriate limits. For example, to compute $\int_\gamma x^2 y dx$, where γ is the circle given by $x(t)+iy(t)=\mathrm{e}^{2\pi i t},\ 0\le t\le 1$, we may write

$$\begin{aligned}\int x^2 y\,dx &= \int_0^1 \cos^2(2\pi t)\sin(2\pi t)\frac{d}{dt}(\cos 2\pi t)\,dt \\ &= -2\pi\int_0^1 \cos^2(2\pi t)\sin^2(2\pi t)\,dt = -(2\pi)(1/8) = -\pi/4.\end{aligned}$$

Note: Stokes' theorem for the plane, which we have quoted above, follows at once from the version with which students are familiar from vector calculus. To see this, put $\mathbf{A} = Q(x,y)\mathbf{e_1} + P(x,y)\mathbf{e_2} + 0\mathbf{e_3}$ and apply Stokes' theorem in the form
$\int_{\mathcal{A}} \mathrm{curl}\,\mathbf{A}.\mathbf{n}\,d\mathcal{A} = \int_\gamma \mathbf{A}.\mathbf{dr}$ with $\mathbf{n}=\mathbf{e_3}$.

Returning to the proof of Cauchy's theorem we write $f(z)=u(x,y)+iv(x,y),\ z=x+iy$. In order to prove Cauchy's therem using Stokes' theorem we have to assume that the derivative $f'(z)$ is continuous. It follows that the first order partial derivatives of u and v with respect to x,y are continuous. We have

$$\int_\gamma f(z)\,dz = \int (u(x(t),y(t)) + iv(x(t),y(t)))(x'(t)+iy'(t))\,dt,$$

the integral being taken between the relevant values of t. Multiplying out the brackets and bearing in mind the above comments we obtain

$$\int_\gamma f(z)\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy).$$

Now apply Stokes' theorem to transform the line integrals. (put $Q=u, P=-v$ in the first of the line integrals; put $Q=v, P=u$ in the second) We obtain

$$\int_\gamma f(z)\,dz = \int_{\mathcal{A}}\Big(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big)\,dx\,dy + i\int_{\mathcal{A}}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big)\,dx\,dy.$$

(this depends on the assumed continuity of the first order partial derivatives of u,v)

Both double integrals are zero by virtue of the Cauchy-Riemann equations and Cauchy's theorem is proved.

Pure mathematicians are unimpressed by this proof! Its main weakness is the fact that it *assumes* that $f'(z)$ is continuous — which is not required in better proofs of the theorem. Moreover, Stokes' theorem is not easy to prove rigorously and it is perhaps no harder to give a rigorous proof of Cauchy's theorem (without the assumption that f' is continuous) than it is to prove Stokes' theorem rigorously. Anyway, the aim of this course is to familiarise students with the basic ideas of complex variable theory and we shall not be deterred from proceeding to derive further results using Cauchy's theorem. Nor shall we hesitate to employ diagramatic proofs where this is appropriate.

As a corollary we prove the following result.

Theorem 6.3 *If a domain D is bounded externally by a simple smooth closed curve γ_1 (strictly by* tr γ_1*) and internally by a simple smooth closed curve γ_2, and $f(z)$ is analytic in a domain containing D and its boundary, then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz,$$

where both intgrals are taken in the same sense, conventionally the positive (anti-clockwise) sense.

Referring to the diagram we connect tr γ_1 and tr γ_2 by a straight line 'slit', as indicated. Applying Cauchy's theorem to the contour

$$A \to B \to C \to D \to E \to F \to G \to H \to A$$

and noting (in an obvious notation) that

$$\int_{B\to C} f(z)\,dz = -\int_{F\to G} f(z)\,dz$$

(because the line segment $B \to C$ is the reverse of the line segment $F \to G$) we obtain

$$\int_{\gamma_1} f(z)\,dz + \int_{\widetilde{\gamma_2}} f(z)\,dz = 0$$

so that

$$\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz = 0$$

and the stated result follows.

A similar argument establishes the following generalisation of this result.

Theorem 6.4 *If a domain D is bounded externally by a simple smooth closed curve γ and internally by simple smooth closd curves $\gamma_1, \gamma_2, \ldots, \gamma_n$, as indicated in the diagram, and if f is analytic in a domain which contains D and its boundary then*

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz.$$

6.2 Cauchy's integral formula

The following important result is due to Cauchy.

Theorem 6.5 *(Cauchy's integral formula) Let γ be a simple smooth closed curve and suppose that f is analytic in a domain containing* $\operatorname{tr}\gamma$ *and its interior $\mathcal{D}$. Then if $z \in \mathcal{D}$*

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

the integral being taken in the positive (anti-clockwise)sense.

Since $z \in \mathcal{D}$ $\exists \rho > 0$ such that $\overline{N_\rho}(z) \subset \mathcal{D}$) Let Γ denote the circle centre z and radius ρ.

Referring to the diagram we note that $f(\zeta)/(\zeta - z)$ is an analytic function of ζ in the hatched region and applying theorem 6.3 we conclude that

$$\int_{\gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

both integrals being taken in the positive sense. Now

$$\int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{\Gamma} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + f(z)\int_{\Gamma} \frac{d\zeta}{\zeta - z} = 2\pi i f(z) + \int_{\Gamma} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta$$

by virtue of an argument which is virtually identical to that of Example 5.5 of Chapter 5 — one puts $\zeta = z + \rho\,\mathrm{e}^{i\theta},\ 0 \le \theta \le 2\pi$ to parametrise Γ. (If you prefer you can use $\zeta = z + \rho\,\mathrm{e}^{2\pi i t},\ 0 \le t \le 1$ — it amounts to the same thing)

Now

$$\left|\int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta\right| \le \int_0^{2\pi} \frac{|f(\zeta) - f(z)|}{\rho}|\zeta'(\theta)|\,d\theta$$

(we've used theorem 5.2)

Since f is continuous at z, given $\epsilon > 0\ \exists\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ whenever $|\zeta - z| < \delta$. Choosing $\rho < \delta$ we see that

$$\left|\int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta\right| < \int_0^{2\pi} \frac{\epsilon|i\rho\,\mathrm{e}^{i\theta}|}{\rho}\,d\theta = 2\pi\epsilon.$$

This argument applies for every $\epsilon > 0$. It follows that

$$2\pi i f(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, \quad f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta,$$

as required.

It is possible to deduce from Cauchy's integral formula that f is infinitely differentiable at z and that the derivative of f to all orders can be computed by formally differentiating with respect to z under the integral sign. Thus

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \tag{6.1}$$

Now for some examples.

Example 6.1 *Evaluate $\int_\gamma \frac{\mathrm{e}^z}{z}\,dz$, where $\gamma(\theta) = \mathrm{e}^{i\theta},\ 0 \le \theta \le 2\pi$, by using Cauchy's integral formula. Deduce the values of two real integrals.*

Let $f(z) = \mathrm{e}^z$. Then f is analytic in $\mathbf{C}$ and we may apply Cauchy's integral formula in the form

$$f(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - 0}\,dz$$

(where we've used z as the integration variable rather that ζ)

It follows that

$$1 = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - 0}\,dz, \quad \int_\gamma \frac{\mathrm{e}^z}{z}\,dz = 2\pi i.$$

Using the given parametrisation of γ we have $dz = i\,\mathrm{e}^{i\theta}\,d\theta$ and

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{e}^{i\theta}}\,\frac{i\,\mathrm{e}^{i\theta}\,d\theta}{\mathrm{e}^{i\theta}} = 2\pi i.$$

This gives

$$\int_0^{2\pi} e^{\cos\theta + i\sin\theta}\, d\theta = 2\pi, \quad \int_0^{2\pi} e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))\, d\theta = 2\pi.$$

Equating real and imaginary parts we obtain the results

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\, d\theta = 2\pi, \quad \int_0^{2\pi} e^{\cos\theta}\sin(\sin\theta)\, d\theta = 0.$$

Example 6.2 *Evaluate*

$$\int_\gamma \frac{e^z}{(z-1)(z-3)}\, dz,$$

taken round the circle γ given by $|z| = 2$ in the positive (anti-clockwise) sense. What is the value of the integral taken arround the circle $|z| = 1/2$ in the positve sense?

Put $f(z) = e^z/(z-3)$. Then f is analytic in a domain which contains the circle $|z| = 2$ and its interior (but not, of course, the point $z = 3$). Cauchy's integral formula is applicable and we have

$$f(1) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - 1)}\, d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{e^\zeta}{(\zeta-1)(\zeta-3)}\, d\zeta.$$

We conclude that

$$\int_\gamma \frac{e^\zeta}{(\zeta-1)(\zeta-3)}\, d\zeta = 2\pi i f(1) = -\pi\, e\, i.$$

By Cauchy's theorem the integral taken round the circle $|z| = 1/2$ in the positive sense is zero because the integrand is analytic in a domain which contains the circle and its interior.

6.3 Laurent's theorem

We are now in a position to prove Laurent's theorem which has great theoretical importance.

Theorem 6.6 *(Laurent's[2] theorem) Suppose that $f(z)$ is analytic in the annulus $D = \{z : R_1 < |z - a| < R_2\}$. Then for all $z \in D$*

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-a)^n,$$

where

$$A_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta,$$

and γ is any circle with centre a and radius R such that $R_1 < R < R_2$.

[2]Pierre Alphonse Laurent. This theorem appeared in 1843

We give an outline of the proof.

For $z \in D$ choose ρ_1 and ρ_2 such that $R_1 < \rho_1 < |z-a| < \rho_2 < R_2$.

Let $\gamma_1,\ \gamma_2$ be the circles centre a given by $\{\zeta : |\zeta - a| = \rho_j,\ j = 1, 2\}$ and parametrised by
$\zeta = \gamma_j(\theta) = a + \rho_j\,\mathrm{e}^{i\theta},\ 0 \le \theta \le 2\pi,\ j = 1, 2.$

Referring to the diagram we connect tr γ_1 and tr γ_2 by a straight line slit (after the fashion of theorem 6.3) and apply Cauchy's integral formula to the contour shown. This gives (since the contributions form the slit, being taken in opposite senses, cancel)

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\tilde{\gamma_1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta \qquad (6.2)$$

We aim to expand the two integrals on the right-hand side of equation 6.2 in powers of $(z-a)$. To this end consider first

$$\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)\,d\zeta}{(\zeta - a)[1 - \frac{(z-a)}{(\zeta-a}]} \qquad (6.3)$$

Now, for any $\alpha \in \mathbf{C},\ \alpha \neq 1$,

$$1+\alpha+\alpha^2+\cdots+\alpha^{n-1}=\frac{1-\alpha^n}{1-\alpha} \tag{6.4}$$

so that

$$\frac{1}{1-\alpha}=1+\alpha+\alpha^2+\cdots+\alpha^{n-1}+\frac{\alpha^n}{1-\alpha}.$$

We apply this formula to the integral in equation 6.3 with $\alpha=(z-a)/(\zeta-a)$ to obtain

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-a)}\Big(\sum_{n=0}^{n_1}\frac{(z-a)^n}{(\zeta-a)^n}\Big)\,d\zeta+R_{n_1},$$

where

$$R_{n_1}=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)(z-a)^{n_1+1}}{(\zeta-a)(\zeta-a)^{n_1+1}}\frac{1}{(\zeta-z)/(\zeta-a)}\,d\zeta,$$

since, with our choice of α, $1-\alpha=(\zeta-z)/(\zeta-a)$. Thus,

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\sum_{n=0}^{n_1}A_n(z-a)^n+R_{n_1},$$

where

$$A_n=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta,\quad R_{n_1}=\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(\zeta)(z-a)^{n_1+1}}{(\zeta-a)^{n_1+1}}\frac{1}{(\zeta-z)}\,d\zeta,$$

Note that the integrand in the equation defining A_n is analytic in the annulus D and by theorem 6.3 A_n may be calculated by integrating round any circle γ centre a and radius R, where $R_1<R<R_2$, as stated in the ennunciation of Laurent's theorem.

We now examine the behaviour of R_{n_1} as $n_1\to\infty$. We parametrise γ_2 by $\zeta=\gamma_2(\theta)=a+\rho_2\,\mathrm{e}^{i\theta},\ 0\le\theta\le 2\pi$. We then obtain, using Theorem 5.2 of Chapter 5 that

$$|R_{n_1}|\le\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(\zeta)|\rho_2|z-a|^{n_1+1}\,d\theta}{{\rho_2}^{n_1+1}|\zeta-z|}.$$

Since f is continuous $|f|$ is bounded on the circle γ_2 and $\exists M\in\mathbf{R}$ such that $|f(\zeta)|\le M,\ \zeta\in\operatorname{tr}\gamma_2$. Also, $|\zeta-z|=|(\zeta-a)-(z-a)|\ge||\zeta-a|-|z-a||=\rho_2-|z-a|$. It follows that

$$|R_{n_1}|\le\frac{\rho_2}{2\pi}\int_0^{2\pi}\frac{M}{(\rho_2-|z-a|)}\frac{|z-a|^{n_1+1}}{{\rho_2}^{n_1+1}}\,d\theta=\frac{M\rho_2}{(\rho_2-|z-a|)}\Big(\frac{|z-a|}{\rho_2}\Big)^{n_1+1}\to 0$$

as $n_1\to\infty$ since $|z-a|/\rho_2<1$. Consequently, $R_{n_1}\to 0$ as $n_1\to\infty$.

Returning to equation 6.2 we consider the second integral on the right hand side of this equation, namely

$$-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{z-\zeta}\,d\zeta=\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)\,d\zeta}{(z-a)[1-\frac{(\zeta-a)}{(z-a}]}$$

By an application of formula 6.4, this time with $\alpha = (\zeta - a)/(z - a)$, we obtain

$$-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta - z}\,d\zeta == \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(z-a)}\Big(\sum_{n=0}^{n_2}\frac{(\zeta-a)^n}{(z-a)^n}\Big)\,d\zeta + S_{n_2} \qquad (6.5)$$

where

$$S_{n_2} = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)(\zeta-a)^{n_2+1}}{(z-a)^{n_2+1}}\frac{1}{(z-\zeta)}\,d\zeta.$$

We repeat the argument developed above.

Since f is continuous $|f|$ is bounded on the circle γ_1 and $\exists N \in \mathbf{R}$ such that $|f(\zeta)| \le N,\ \zeta \in \text{tr}\ \gamma_1$. Also, $|\zeta - z| = |(\zeta - a) - (z - a)| \ge ||\zeta - a| - |z - a|| = |z - a| - \rho_1$. It follows that

$$|S_{n_2}| \le \frac{\rho_1}{2\pi}\int_0^{2\pi}\frac{N}{(|z-a|-\rho_1)}\frac{{\rho_1}^{n_2+1}}{|z-a|^{n_2+1}}\,d\theta = \frac{N\rho_1}{(|z-a|-\rho_1)}\Big(\frac{\rho_1}{|z-a|}\Big)^{n_2+1} \to 0 \text{ as } n_2 \to \infty$$

since $\rho_1/|z - a| < 1$. Consequently, $S_{n_2} \to 0$ as $n_2 \to \infty$. Referring to equation 6.5 we may write, making the change of variable $m = -(n + 1)$ in the sum

$$\begin{aligned} -\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta - z}\,d\zeta &= \sum_{m=-(n_2+1)}^{-1}\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{(\zeta-a)^{m+1}}(z-a)^m + S_{n_2} \\ &= \sum_{m=-(n_2+1)}^{-1} A_m(z-a)^m + S_{n_2} \end{aligned}$$

where $S_{n_2} \to 0$ as $n_2 \to \infty$. and A_m is defined in the ennunciation of Laurent's theorem.

We now deduce from equation 6.2 that

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n,$$

where

$$A_n = \frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta,$$

and γ is any circle with centre a and radius R such that $R_1 < R < R_2$.

$\sum_{-\infty}^{\infty}$ is to be understood as the limit of $\sum_{-p}^{q}$ as q and p tend to infinity, *independently*; this should be clear from the proof.

Laurent's theorem has great theoretical significance and we shall use it in our development of the residue calculus in the next chapter. Meanwhile the following is an interesting example.

Example 6.3 *Apply Laurent's theorem to expand the function* $e^{\frac{x}{2}(z-1/z)}$ *in the form*

$$e^{\frac{x}{2}(z-1/z)} \sum_{n=-\infty}^{\infty} J_n(x) z^n$$

and discuss some of the properties of the functions $J_n(x)$*, showing in particular that they satisfy Bessel's differential equation. (See equation 13.1 of Chapter 13 where Bessel's equation arises in the context of finding solutions of the 3-dimensional Laplace equation in cylindrical polar coordinates)*

The functions $J_n(x)$ are known as the Bessel functions of order n and are important in mathematical physics.
Note that $e^{\frac{x}{2}(z-1/z)}$ is analytic in $\mathbf{C} - \{0\}$ and Laurent's theorem (about the origin, $a = 0$) gives

$$e^{\frac{x}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(x) z^n \quad \forall z \in \mathbf{C} - \{0\} \tag{6.6}$$

where

$$J_n(x) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^{\frac{x}{2}(\zeta-1/\zeta)}\, d\zeta}{\zeta^{n+1}}$$

and γ_1 is the circle centre the origin and radius 1 parametrised by $\zeta(\theta) = e^{i\theta},\ -\pi \le \theta \le \pi$.

We obtain

$$\begin{aligned} J_n(x) &= \frac{1}{2\pi i} \int_{\gamma_1} \frac{e^{\frac{x}{2}(\zeta-1/\zeta)}\, d\zeta}{\zeta^{n+1}} = \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{ix\sin\theta} (ie^{i\theta}) e^{-i(n+1)\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x\sin\theta - n\theta)}\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{0} e^{i(x\sin\theta - n\theta)}\, d\theta + \frac{1}{2\pi} \int_{0}^{\pi} e^{i(x\sin\theta - n\theta)}\, d\theta \end{aligned}$$

Putting $u = -\theta$ in the first integral gives

$$J_n(x) = \frac{-1}{2\pi} \int_{\pi}^{0} e^{i(-x\sin u + nu)}\, du + \frac{1}{2\pi} \int_{0}^{\pi} e^{i(x\sin\theta - n\theta)}\, d\theta.$$

Re-writing with u replaced by θ and combining the integrals using $\cos\alpha = (e^{i\alpha} + e^{-i\alpha})/2$ gives

$$J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta, \quad \text{Bessel's integral for the Bessel function } J_n(x).$$

It is clear that the left hand side of equation 6.6 is invariant under $z \mapsto -1/z$ and

$$e^{\frac{x}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(x) z^n = \sum_{n=-\infty}^{\infty} J_n(x) (-1)^n z^{-n} \quad \forall z \in \mathbf{C} - \{0\} \tag{6.7}$$

For any positive integer m we equate coefficients of z^m to obtain $J_{-m}(x) = (-1)^m J_m(x)$.

Next, expressing the exponential in equation 6.6 as a product of exponentials we obtain

$$e^{\frac{x}{2}(z-1/z)} = \left(\sum_{s=0}^{\infty} \frac{\left(\frac{xz}{2}\right)^s}{s!}\right)\left(\sum_{r=0}^{\infty} \frac{\left(\frac{-x}{2z}\right)^r}{r!}\right) = \sum_{n=-\infty}^{\infty} J_n(x) z^n \quad \forall z \in \mathbf{C} - \{0\} \tag{6.8}$$

Since the two series are absolutely convergent we can multiply them in the obvious way. We note that the term in z^n in the product arises from the values of r and s such that $s-r=n$. Picking out the terms in the product for which $s=r+n$ and letting r go from 0 to ∞ we see that the coefficient of z^n in the product is

$$\sum_{r=0}^{\infty}\left(\frac{x}{2}\right)^{r+n}(-1)^r\left(\frac{x}{2}\right)^r \frac{1}{(n+r)!}\frac{1}{r!}$$

We therefore obtain

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r} \tag{6.9}$$

It is a simple calculation to verify what is already implicit from equation 6.6, that the series for J_n converges for all real or complex x.

If we differentiate equation 6.6 with repect to x and z in turn we obtain

$$\frac{1}{2}(z-1/z)\sum_{n=-\infty}^{\infty} J_n(x) z^n = \sum_{n=-\infty}^{\infty} {J_n}'(x) z^n \tag{6.10}$$

$$\frac{x}{2}(1+1/z^2)\sum_{n=-\infty}^{\infty} J_n(x) z^n = \sum_{n=-\infty}^{\infty} nJ_n(x) z^{n-1} \tag{6.11}$$

Equating the coefficients of z^n in the first of these equations and of z^{n-1} in equation 6.11 we obtain

$${J_n}'(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)), \quad \frac{x}{2}(J_{n-1}(x) + J_{n+1}(x)) = nJ_n(x) \tag{6.12}$$

$$2{J_n}'(x) = J_{n-1}(x) - J_{n+1}(x), \quad \frac{2nJ_n(x)}{x} = J_{n-1}(x) + J_{n+1}(x) \tag{6.13}$$

Adding and subtracting the two equations 6.13 immediately gives

$$x{J_n}'(x) + nJ_n(x) = xJ_{n-1}(x) \tag{6.14}$$

$$x{J_n}'(x) - nJ_n(x) = -xJ_{n+1}(x) \tag{6.15}$$

Differentiating equation 6.14 with respect to x gives

$$\begin{aligned} x{J_n}''(x) + (n+1){J_n}'(x) = xJ'_{n-1}(x) + J_{n-1} &= (n-1)J_{n-1}(x) - xJ_n(x) + J_{n-1}(x) \\ &= nJ_{n-1}(x) - xJ_n(x) \end{aligned}$$

using equation 6.15 with n replaced by $n-1$. Expressing J_{n-1} in terms of the J_n using equation 6.14 gives

$$x{J_n}''(x) + (n+1){J_n}'(x) = n((nJ_n(x))/x + {J_n}'(x)) - xJ_n(x), \quad x^2{J_n}'' + x{J_n}' + (x^2-n^2)J_n = 0,$$

so that J_n satisfies Bessel's differential equation, as required.

The defining relation for the Bessel functions gives

$$e^{\frac{x+y}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(x+y)z^n \quad \forall z \in \mathbf{C} - \{0\}.$$

Writing the left hand side as the product of two exponentials and then replacing them in terms of Bessel functions immediately gives

$$\left(\sum_{r=-\infty}^{\infty} J_r(x)z^r\right)\left(\sum_{s=-\infty}^{\infty} J_s(y)z^n\right) = \sum_{n=-\infty}^{\infty} J_n(x+y)z^n.$$

Equating powers of z^n on both sides yields

$$J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x)J_{n-r}(y).$$

Note: We state without proof that if an analytic function f has a Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$, valid in some domain D, then the coefficients A_n are unique. Thus, if it is also true that $f(z) = \sum_{n=-\infty}^{\infty} B_n(z-a)^n$ in D then $A_n = B_n$ for all n.

6.4 Taylor's theorem

We can regard Taylor's[3] theorem as a particular case of Laurent's theorem.

Theorem 6.7 *(Taylor's theorem) If f is analytic in the domain D consisting of all points z such that $|z-a| < R$ then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n, \ \forall z \in D.$$

Laurent's theorem is applicable and we may write

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n,$$

where

$$A_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta,$$

γ being any circle with centre a and radius R_1 such that $R_1 < R$. Since f is analytic it follows from Cauchy's theorem that the A_n are zero for $n = -1, -2, -3, \cdots$

[3]This is named after Brook Taylor (1685-1731) who in 1715 published the real variable version of this theorem

For $n = 0$ $A_n = f(a)$ by Cauchy's integral formula, whilst for $n > 0$ we have

$$A_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta = \frac{1}{n!}\frac{d^n}{da^n}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - a)}\,d\zeta\right) = \frac{f^{(n)}(a)}{n!}.$$

(the formal differentiation under the integral can be rigorously justified)

It now follows that the standard series for the elementary functions which hold in real variable theory also hold for complex variables. For example

$$\mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

the series converging for all complex z. This must be true since the exponential is analytic in $\mathbf{C}$ and therefore R, in our statement of Taylor's theorem, can be taken arbitrarily large.

The standard series for $\sin,\ \cos$ hold for all complex z; there seems to be no point writing down all the details which are identical to the real variable case.

We can say immediately that the Taylor series for $\tan z$ will converge for all z such that $|z| < \pi/2$, including of course real numbers for which this condition is satisfied. It would be very hard to prove this directly in the real variable case by developing the Taylor series since we would require to obtain an estimate of the n-th derivative of $\tan x$ — not a pleasant prospect!

Chapter 7

Calculus of residues

7.1 Singularities

Definition 7.1 *If f is analytic in the domain $D = \{z : 0 < |z - a| < R\}$, but not at $z = a$, we say that a is an isolated singularity of f.*

For example, the function $1/(z-1)$ is analytic everywhere, except at $z = 1$ where it has an isolated singularity.

Following on in the notation of our definition we note that Laurent's theorem applies and we can write

$$f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n + \sum_{n=1}^{\infty} B_n (z-a)^{-n}, \quad B_n = A_{-n}, \ 0 < |z-a| < R,$$

in the notation used previously.

If all the B_n are zero then $f(z) \to A_0$ as $z \to a$. If we re-define $f(a)$ by $f(a) = A_0$ then f is analytic in the domain consisting of the points z such that $|z - a| < R$. In such a situation we say (for obvious reasons) that a is a *removable* singularity.

If all but a finite number of the B_n are zero so that

$$f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n + \sum_{n=1}^{m} B_n (z-a)^{-n}, \quad B_m \neq 0, \ 0 < |z-a| < R,$$

we say that f has a pole of order m at the point $z = a$. If $m = 1$ we say that the pole is *simple.*

Thus $f(z) = 1/(z-1)$ has a simple pole at $z = 1$.

More generally, we have the following result.

Theorem 7.1 *If g is a function which is analytic in a neighbourhood of $z = a$, with $g(a) \neq 0$, then $f(z) = g(z)/(z - a)$ has a simple pole at $z = a$.*

This is not hard to see. Since g is analytic in a neighbourhood of $z = a$ we can write $g(z) = \sum_{n=0}^{\infty} c_n(z - a)^n$, where $c_0 \neq 0$ since $g(a) \neq 0$, the series converging inside some circle centre a and radius $\delta > 0$. We then have

$$f(z) = \frac{c_0}{(z - a)} + c_1 + c_2(z - a) + c_3(z - a)^2 + \cdots$$

as the Laurent expansion of f about $z = a$. Since $c_0 \neq 0$ it is clear that f has a simple pole at $z = a$.

We see, for example, that $\mathrm{e}^z/(z - 4)$ has a simple pole at $z = 4$.

A third possibility arises when the Laurent expansion is such that an infinite number of the coefficients B_n are non-zero. In this case we say that the fuction has an *essential singularity* at $z = a$. An example is given by the function

$$\mathrm{e}^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}$$

which is analytic everywhere, except at $z = 0$ where it has an essential singularity.

7.2 Calculus of residues

Continuing in the above notation, suppose that f has an isolated singularity at $z = a$ so that

$$f(z) = \sum_{n=0}^{\infty} A_n(z - a)^n + \sum_{n=1}^{\infty} B_n(z - a)^{-n}, \quad B_n = A_{-n}, \ 0 < |z - a| < R,$$

for some $R > 0$. For all integers n the coefficients A_n are given by

$$A_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - a)^{n+1}},$$

where γ is any circle centre a which excludes all other singularities of f.

We define the *residue* of f at a to be the coefficient of $1/(z-a)$ in the Laurent expansion; in other words, the residue of f at a is equal to $B_1 = A_{-1}$ or, in terms of the integral representation of B_1, $\frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta$.

Theorem 7.2 *(Cauchy's residue theorem) If f is analytic, except for a finite number of poles in a domain which includes the trace of a closed contour γ and its interior $\mathcal{D}$ then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{\mathcal{R}}$$

where the sum denotes the sum of all the residues of f at poles which lie in $\mathcal{D}$, it being understood that γ does not pass through any singularity of f.

Referring to the diagram we see, from theorem 6.4 of Chapter 6, that

$$\int_\gamma f(z)\,dz = \sum_{r=1}^{n} \int_{\gamma_r} f(z)\,dz,$$

where $a_1, a_2, \ldots, a_n$ are the poles of f inside $\mathcal{D}$ and γ_r denotes a circle, centre a_r and suitably small radius δ_r to guarantee that $\overline{N_{\delta_r}}(a_r) \subset \mathcal{D}$. We then have

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{r=1}^{n} \frac{1}{2\pi i}\int_{\gamma_r} f(z)\,dz = \sum_{\mathcal{R}}$$

and the stated result immediately follows.

This theorem provide a powerful technique for the computation of many integrals which would otherwise be extremely difficult to deal with. In order to use it we have to be able to calculate residues. We now describe some of the standard techniques for the calculation of residues.

Theorem 7.3 *If $f(z)$ has a simple pole at $z = a$ then the residue of f at a is equal to* $\lim_{z\to a}(z-a)f(z)$

To see this we may argue that near $z = a$ $f(z)$ has the form

$$f(z) = \frac{B_1}{z-a} + \phi(z),$$

where ϕ is analytic at $z = a$. We therefore have $B_1 = (z-a)f(z) - (z-a)\phi(z)$ and, proceeding to the limit, we immediately obtain $B_1 = \lim_{z\to a}(z-a)f(z)$, as required.

As a corollary we note that if $\psi(z)$ is analytic at a, and $\psi(a) \neq 0$, then the residue of $(\psi(z))/(z-a)$ is equal to $\psi(a)$. The proof is easy:

We noted previously (see Theorem 7.1) that $(\psi(z))/(z-a)$ has a simple pole at $z = a$ and its residue is therefore equal to $\lim_(z-a)(\psi(z))/(z-a) = \psi(a)$.

The following result is often very useful.

Theorem 7.4 *If $f(z)$ is analytic and non-zero at $z = a$ and if $\psi(z)$ is analytic at $z = a$, with a simple zero there, then the residue of $f(z)/\psi(z)$ is $f(a)/\psi'(a)$.*

First we note that near $z = a$ $\psi(z) = (z-a)\psi_1(z)$, where ψ_1 is analytic and non-zero at $z = a$. (This reflects the fact that ψ is analytic at a and has a simple zero there). It follows at once from Theorem 7.1 that $f(z)/\psi(z)$ has a simple pole at $z = a$.

The residue is

$$\lim_{z\to a}(z-a)\frac{f(z)}{\psi(z)} = \lim_{z\to a}\frac{f(z)}{\frac{\psi(z)-\psi(a)}{(z-a)}} = \frac{f(a)}{\psi'(a)},$$

as required.

Another useful result is the following.

Theorem 7.5 *Suppose that $g(z)$ is analytic at $z = a$. The function f given by $f(z) = g(z)/(z-a)^n$, where $n \geq 1$ is an integer, has residue $g^{(n-1)}(a)/(n-1)!$ at $z = a$.*

The proof is easy. Since g is analytic

$$g(z) = \sum_{r=0}^{\infty}\frac{g^r(a)}{r!}(z-a)^r,$$

near $z = a$. We then have

$$\frac{g(z)}{(z-a)^n} = \sum_{r=0}^{\infty}\frac{g^r(a)}{r!}(z-a)^{r-n}.$$

The required residue is the coefficient of $1/(z-a)$ in this sum and is given by $r = n-1$; the stated result follows at once.

Consider the following examples.

Example 7.1 *The residue of* $\cot z$ *at* $z = k\pi$*, where* k *is an integer, is equal to* 1.

Since $\cot z = (\cos z)/(\sin z)$ we see that since $\sin z$ has a simple zero at $z = k\pi$ (it's simple because the derivative of $\sin z$, namely $\cos z$ is non-zero at $z = k\pi$) $\cot z$ has a simple pole at $z = k\pi$, as follows from theorem 7.4. The same theorem shows that the residue of $\cot z$ at $k\pi$ is equal to $(\cos(k\pi))/(\sin)'(k\pi) = 1$.

Example 7.2 *The residue of* $\mathrm{e}^{iz}/(z^2 + a^2)$ *at* $z = ai$ *is equal to* $\mathrm{e}^{-a}/(2ai)$.

The denominator of this function obviously has a simple zero at $z = ai$. By theorem 7.4 the residue of the function at $z = ai$ is

$$\left.\frac{\mathrm{e}^{iz}}{\frac{d}{dz}(z^2 + a^2)}\right|_{z=ai} = \frac{\mathrm{e}^{-a}}{2ai}$$

as required.

Example 7.3 *Find the residue of* $1/(z^4 + 1)$ *at* $z = \mathrm{e}^{i\pi/4}$.

The denominator of this function equals zero at $z = \mathrm{e}^{i\pi/4}$ and the zero is simple because the derivative of the denominator is non-zero at this point. Theorem 7.4 tells us that the required residue is equal to

$$\left.\frac{1}{\frac{d}{dz}(z^4 + 1)}\right|_{z=\mathrm{e}^{i\pi/4}} = \frac{1}{4}\,\mathrm{e}^{-3\pi i/4} = -\frac{1}{4}\,\mathrm{e}^{i\pi/4} = -\frac{\sqrt{2}}{8}(1 + i).$$

Example 7.4 *Find the residue of* $1/(z^2 + 1)^2$ *at* $z = i$.

The denominator of this function has a double zero at $z = i$ so this time theorem 7.4 isn't applicable. However, we can appeal to theorem 7.5. We note that

$$\frac{1}{(z^2 + 1)^2} = \frac{1}{(z + i)^2}\frac{1}{(z - i)^2}$$

and by theorem 7.5 the residue at $z = i$ is equal to

$$\left.\frac{d}{dz}\left(\frac{1}{(z + i)^2}\right)\right|_{z=i} = \left.\frac{-2}{(z + i)^3}\right|_{z=i} = -\frac{i}{4}.$$

Another way of dealing with this problem is as follows. In principle, we need to compute the Laurent expansion of $1/(z^2+1)^2$ about $z = i$ and the required residue is the coefficient of $1/(z - i)$ in this expansion. To simplify the writing we put $z = i + \zeta$ and pick out the coefficient of $1/\zeta$. We have

$$\frac{1}{(z^2 + 1)^2} = \frac{1}{[(\zeta + i)^2 + 1]^2} = \frac{1}{(\zeta^2 + 2i\zeta)^2} = \frac{1}{(2i\zeta)^2}[1+\zeta/(2i)]^{-2} = -\frac{1}{4\zeta^2}\left[1+\left(\frac{\zeta}{2i}\right)(-2)+\cdots\right]$$

The coefficient of $1/\zeta$ is $-i/4$, as we found above by another apparently different — but equivalent method.

Example 7.5 *Let*

$$Let f(z) = \frac{A_1}{(z-a)} + \frac{A_2}{(z-a)^2} + \cdots + \frac{A_n}{(z-a)^n}.$$

Suppose that $\zeta \neq a$. Show that the residue of $f(z)/(z-\zeta)$ at $z = a$ is $-f(\zeta)$.

We need to pick out the coefficient of $1/(z-a)$ in the Laurent expansion of $f(z)/(z-\zeta)$ about $z = a$. The easiest way to do this is to expand $1/(z-\zeta)$ about $z = a$ using

$$\frac{1}{z-\zeta} = \frac{-1}{(\zeta-a)(1-\frac{z-a}{\zeta-a})} = \frac{-1}{\zeta-a}\sum_{s=0}^{\infty}\Big(\frac{z-a}{\zeta-a}\Big)^s,$$

the geometric series being convergent when z is sufficiently close to a. It follows that

$$\frac{f(z)}{z-\zeta} = \frac{(-1)}{\zeta-a}\Big(\frac{A_1}{(z-a)} + \frac{A_2}{(z-a)^2} + \cdots \frac{A_n}{(z-a)^n}\Big)\sum_{s=0}^{\infty}\Big(\frac{z-a}{\zeta-a}\Big)^s.$$

Picking out the coefficient of $1/(z-a)$ we obtain the residue of $f(z)/(z-\zeta)$ at $z = a$ as

$$-\Big(\frac{A_1}{(\zeta-a)} + \frac{A_2}{(\zeta-a)^2} + \cdots \frac{A_n}{(\zeta-a)^n}\Big) = -f(\zeta).$$

We conclude by showing how the residue theorem provides a powerful method for the computation of certain integrals using a technique often referred to as *contour integration.*

7.3 Contour integration

In what follows let C_R denote the contour consisting of the portion of the real axis between $z = -R$ and $z = +R$, together with the semi-circle γ_R, where $\gamma_R(\theta) = Re^{i\theta},\ 0 \leq \theta \leq \pi$.

Example 7.6 *By considering*

$$\int_{C_R} \frac{e^{iz}}{z-ai}\,dz,\ \ a > 0,$$

show that

$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx = 2\pi\,e^{-a}\,.$$

Let $f(z) = \mathrm{e}^{iz}/(z - ai)$. f has a simple pole at $z = ai$ with residue

$$\lim_{z \to ai} (z - ai) \frac{\mathrm{e}^{iz}}{z - ai} = \mathrm{e}^{-a}.$$

Application of Cauchy's residue theorem to the contour C_R then gives

$$\int_{-R}^{0} \frac{\mathrm{e}^{ix}}{x - ai}\, dx + \int_{0}^{R} \frac{\mathrm{e}^{ix}}{x - ai}\, dx + \int_{\gamma_R} \frac{\mathrm{e}^{iz}}{z - ai}\, dz = 2\pi i\, \mathrm{e}^{-a}.$$

Changing the variable to $-x$ in the first integral leads to

$$\int_{0}^{R} \frac{x(\mathrm{e}^{ix} - \mathrm{e}^{-ix}) + ai(\mathrm{e}^{ix} + \mathrm{e}^{-ix})}{x^2 + a^2} + \int_{\gamma_R} f(z)\, dz = 2\pi i\, \mathrm{e}^{-a}.$$

Assuming that $\int_{\gamma_R} f(z)\, dz \to 0$ as $R \to \infty$ we obtain

$$\int_{0}^{\infty} \frac{x \sin x + a \cos x}{x^2 + a^2}\, dx = \pi\, \mathrm{e}^{-a},$$

which is equivalent to the stated result. In order to see that $\int_{\gamma_R} f(z)\, dz \to 0$ as $R \to \infty$ we can argue as follows:

$$\left| \int_{\gamma_R} f(z)\, dz \right| = \left| \int_{0}^{\pi} \frac{\mathrm{e}^{i[R\cos\theta + iR\sin\theta]}\, Ri\, \mathrm{e}^{i\theta}\, d\theta}{R\, \mathrm{e}^{i\theta} - ai} \right| \le \int_{0}^{\pi} \frac{R\, \mathrm{e}^{-R\sin\theta}\, d\theta}{|R\, \mathrm{e}^{i\theta} - ai|}.$$

Using the inequality $||\alpha| - |\beta|| \le |\alpha - \beta|$ we obtain

$$|R\, \mathrm{e}^{i\theta} - ai| \ge ||R\, \mathrm{e}^{i\theta}| - |ai|| = |R - a| = R - a,$$

(the last step assuming $R > a$). We therefore obtain

$$\left| \int_{\gamma_R} f(z)\, dz \right| \le \int_{0}^{\pi} \frac{R}{R - a} e^{-R\sin\theta}\, d\theta = \frac{2R}{R - a} \int_{0}^{\pi/2} e^{-R\sin\theta}\, d\theta.$$

Since $\sin\theta \ge 2\theta/\pi\ \forall \theta \in [0, \pi/2]$ we obtain

$$\left| \int_{\gamma_R} f(z)\, dz \right| \le \frac{2R}{R - a} \int_{0}^{\pi/2} \mathrm{e}^{-2R\theta/\pi}\, d\theta = \frac{\pi}{R - a}(1 - \mathrm{e}^{-R}) \to 0 \text{ as } R \to \infty.$$

Example 7.7 *Let $m > 0,\ a > 0$. Prove, by considering*

$$\int_{C_R} z\, \mathrm{e}^{imz}/(z^4 + a^4)\, dz,$$

that

$$\int_{0}^{\infty} \frac{x \sin mx}{x^4 + a^4}\, dx = \frac{\pi}{2a^2}\, \mathrm{e}^{-\frac{ma}{\sqrt{2}}} \sin\left(\frac{ma}{\sqrt{2}}\right).$$

Let $f(z) = z\,\mathrm{e}^{imz}/(z^4 + a^4),\ a > 0,\ m > 0$ and consider $\int_{C_R} f(z)\,dz$. f has simple poles where $z^4 + a^4 = 0$ i.e. when $z = z_k$, where $z_k = a\,\mathrm{e}^{i(2k+1)\pi/4},\ k = 0, 1, 2, 3$. Only the poles at $z = z_0$ and $z = z_1$ lie inside the contour C_R. Note that $z_1 = iz_0$. The sum of the residues of the poles at $z = z_0,\ z = z_1$ is

$$\frac{z_0\,\mathrm{e}^{imz_0}}{\frac{d}{dz}(z^4 + a^4)\big|_{z=z_0}} + \frac{z_1\,\mathrm{e}^{imz_1}}{\frac{d}{dz}(z^4 + a^4)\big|_{z=z_1}} = \frac{z_0\,\mathrm{e}^{imz_0}}{4{z_0}^3} + \frac{z_1\,\mathrm{e}^{imz_1}}{4{z_1}^3} = \frac{\mathrm{e}^{imz_0}}{4{z_0}^2} + \frac{\mathrm{e}^{imz_1}}{4{z_1}^2}$$

Using $z_1 = iz_0,\ z_0 = ae^{i\pi/4}$ this simplifies to

$$\frac{\mathrm{e}^{-(ma/\sqrt{2})}\sin(ma/\sqrt{2})}{2a^2}.$$

Cauchy's residue theorem applied to C_R gives

$$\int_{-R}^{0} \frac{x\,\mathrm{e}^{ix}}{x^4 + a^4}\,dx + \int_{0}^{R} \frac{x\,\mathrm{e}^{ix}}{x^4 + a^4}\,dx + \int_{\gamma_R} f(z)\,dz = \frac{\pi i\,\mathrm{e}^{-(ma/\sqrt{2})}\sin(ma/\sqrt{2})}{a^2}.$$

Following the procedure adopted in example 7.6 we change the variable from x to $-x$ and let $R \to \infty$ (the integral round γ_R tends to zero – same sort of proof as in example 7.6). This gives

$$\int_{0}^{\infty} \frac{x\sin mx}{x^4 + a^4}\,dx = \frac{\pi\,\mathrm{e}^{-(ma/\sqrt{2})}\sin(ma/\sqrt{2})}{2a^2} \tag{7.1}$$

If one requires to compute

$$\int_{0}^{\infty} \frac{x^2\cos mx}{x^4 + a^4}\,dx$$

this can be accomplished by differentiating equation 7.1 with respect to m. (the differentiation under the integral is valid in this case)

Example 7.8 *By considering*

$$\int_{C_R} \frac{\mathrm{Log}\,(1 - iz)}{z^2 - 2z\sin\alpha + 1}\,dz, \quad 0 \le \alpha < \pi/2,$$

where Log *is the principal value of the logarithm, show that for* $0 \le \alpha < \pi/2$

$$\int_{-\infty}^{\infty} \frac{\arctan x\,dx}{(x^2 - 2x\sin\alpha + 1)} = \frac{\pi\alpha}{2\cos\alpha}, \quad \int_{-\infty}^{\infty} \frac{\ln(1 + x^2)\,dx}{(x^2 - 2x\sin\alpha + 1)} = \frac{2\pi\ln(2\cos(\alpha/2))}{\cos\alpha}.$$

Let

$$f(z) = \frac{\mathrm{Log}\,(1 - iz)}{z^2 - 2z\sin\alpha + 1}.$$

f has a branch point due to the logarithm where $1 - iz = 0$ i.e. where $z = -i$. However, $z = -i$ does not lie inside the contour C_R and we therefore have no difficulty with "multi-valued" logarithms. On the other hand, f has poles where $z^2 - 2z\sin\alpha + 1 = 0$ i.e. where

$z = \sin\alpha \pm i\cos\alpha$ or, equivalently, $z = i\,\mathrm{e}^{-i\alpha},\ z = -i\,\mathrm{e}^{i\alpha}$. Of these, only $z = i\,\mathrm{e}^{-i\alpha}$ lies inside the contour C_R. The residue of $f(z)$ at this pole is

$$\lim_{z\to i\,\mathrm{e}^{-i\alpha}} (z - i\,\mathrm{e}^{-i\alpha}) \frac{\mathrm{Log}\,(1-iz)}{(z - i\,\mathrm{e}^{-i\alpha})(z + i\,\mathrm{e}^{i\alpha})} = \frac{\mathrm{Log}\,(1+\mathrm{e}^{-i\alpha})}{2i\cos\alpha}.$$

Now

$\mathrm{Log}\,(1+\mathrm{e}^{-i\alpha}) = \mathrm{Log}\ \mathrm{e}^{-i\alpha/2}(e^{i\alpha/2}+\mathrm{e}^{-i\alpha/2}) = \mathrm{Log}\ \mathrm{e}^{-i\alpha/2}(2\cos(\alpha/2)) = \ln((2\cos(\alpha/2))+i(-\alpha/2)$

(using the principal value of the logarithm). The above residue is therefore equal to

$$-i\frac{\ln(2\cos(\alpha/2)) + i(-\alpha/2)}{2\cos\alpha}.$$

Applying Cauchy's residue theorem to the contour C_R we immediately obtain

$$\int_{-R}^{R} \frac{\mathrm{Log}\,(1-ix)\,dx}{x^2 - 2x\sin\alpha + 1} + \int_{\gamma_R} f(z)\,dz = \pi\frac{\ln(2\cos(\alpha/2)) + i(-\alpha/2)}{\cos\alpha}.$$

One can again prove, along the lines of the argument presented in example 7.6, although it is slightly more complicated because of the nature of the integrand, that the integral round $\gamma_R \to 0$ as $R\to\infty$. Letting $R\to\infty$ and noting that
$\mathrm{Log}\,(1-ix) = \ln(1+x^2)^{1/2} - i\arctan x$ (remember, we're using the principal value of the logarithm) we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\ln(1+x^2)\,dx}{x^2 - 2x\sin\alpha + 1} &= \frac{2\pi}{\cos\alpha}\ln(2\cos(\alpha/2)), \quad 0\le\alpha<\pi/2, \\ \int_{-\infty}^{\infty} \frac{\arctan x\,dx}{x^2 - 2x\sin\alpha + 1} &= \frac{\pi\alpha}{2\cos\alpha}, \quad 0\le\alpha<\pi/2 \end{aligned}$$

Example 7.9 *By integrating* $\mathrm{e}^{\alpha z}/\cosh\pi z$ *round the rectangle whose sides are* $x = \pm R,\ y = 0,\ y = 1$ *show that*

$$\int_0^\infty \frac{\cosh\alpha x}{\cosh\pi x}\,dx = \frac{1}{2}\sec(\alpha/2), \quad |\alpha| < \pi.$$

Let $f(z) = \mathrm{e}^{\alpha z}/\cosh\pi z$. f has simple poles where $\cosh\pi z = 0$ i.e. where $\cos i\pi z = 0$ i.e. where $z = -i(k+\frac{1}{2}),\ k = 0, \pm1, \pm2, \cdots$ Of these poles, only the one at $z = i/2$ actually lies inside the given rectangular contour. We note that

$$\mathrm{Res}\frac{\mathrm{e}^{\alpha z}}{\cosh \pi z}\Big|_{z=i/2} = \frac{\mathrm{e}^{i\alpha/2}}{\pi \sinh(i\pi/2)} = -\frac{i\,\mathrm{e}^{i\alpha/2}}{\pi}$$

because $\sinh(i\pi/2) = i\sin(\pi/2) = i$.

Application of the residue theorem now gives

$$\int_{-R}^{R} \frac{\mathrm{e}^{\alpha x}\,dx}{\cosh \pi x} + I_1 + \int_{R}^{-R} \frac{\mathrm{e}^{\alpha(x+i)}\,dx}{\cosh \pi(x+i)} + I_2 = 2\pi i\Big(-\frac{i\,\mathrm{e}^{i\alpha/2}}{\pi}\Big) = 2\,\mathrm{e}^{i\alpha/2},$$

where

$$I_1 = \int_0^1 \frac{\mathrm{e}^{\alpha(R+iy)}\,i\,dy}{\cosh \pi(R+iy)}, \quad I_2 = \int_1^0 \frac{\mathrm{e}^{\alpha(-R+iy)}\,i\,dy}{\cosh \pi(-R+iy)}.$$

Assuming for a moment that I_1 and I_2 both tend to zero as $R \to \infty$ (when $|\alpha| < \pi$) we obtain (since $\cosh(\pi(x+i)) = -\cosh \pi x$ — check this)

$$(1+\mathrm{e}^{i\alpha}) \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\alpha x}}{\cosh \pi x}\,dx = 2\,\mathrm{e}^{i\alpha/2}, \text{ so that}$$

$$\cos(\alpha/2) \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\alpha x}}{\cosh \pi x}\,dx = 1, \quad \int_0^{\infty} \frac{\cosh \alpha x}{\cosh \pi x}\,dx = \frac{1}{2}\sec(\alpha/2),\ |\alpha| < \pi.$$

(the last step involves splitting the range of integration from 0 to ∞ and $-\infty$ to 0 and using the change of variable $x \mapsto -x$ in one of the resulting integrals.)

A simple change of variable immediately gives the well known result

$$\int_0^{\infty} \frac{\cosh \lambda x}{\cosh x}\,dx = \frac{1}{2}\pi \sec(\lambda\pi/2),\ |\lambda| < 1.$$

To make the argument water-tight we need to show that I_1 and I_2 both tend to zero as $R \to \infty$. We deal with I_1; I_2 can be treated in an identical fashion. We have

$$|I_1| = \left|\int_0^1 \frac{\mathrm{e}^{\alpha(R+iy)}\,i\,dy}{\cosh \pi(R+iy)}\right| \le \int_0^1 \frac{\mathrm{e}^{\alpha R}\,dy}{|\cosh(\pi i y + \pi R)|}$$

Now

$$\begin{aligned}
|\cosh(\pi i y + \pi R)| &= |\cos \pi y \cosh \pi R + i \sin \pi y \sinh \pi R| \\
&= (\cos^2\pi y \cosh^2\pi R + \sin^2\pi y \sinh^2\pi R)^{1/2} \\
&= (\cosh^2\pi R - \sin^2\pi y)^{1/2} \\
&= ((\cosh^2\pi R)/4 + 3(\cosh^2\pi R)/4 - \sin^2\pi y)^{1/2} \\
&> ((\cosh^2\pi R)/4)^{1/2} = \frac{1}{2}\cosh \pi R \text{ (for all large } R) \\
&> \frac{1}{4}\,\mathrm{e}^{\pi R} \text{ (for all large } R)
\end{aligned}$$

It follows that

$$|I_1| < \int_0^1 \frac{e^{\alpha R}\, dy}{(e^{\pi R})/4} = 4\, e^{(\alpha-\pi)R} \to 0 \text{ as } R \to \infty \quad (\text{since } |\alpha| < \pi)$$

Likewise $I_2 \to 0$ as $R \to \infty$, and the above result is established.

Chapter 8

Origin of some PDE s of Mathematical Physics

In this section we attempt to show how the partial differential equations which we study later arise in a natural way in mathematical physics.

8.1 Notation

We assume that space is Euclidean. Let $Oxyz$ be a rectangular coordinate system so that the position vector $\mathbf{r}$ of a typical point P may be expressed as

$$\mathbf{r} = x\mathbf{e_1} + y\mathbf{e_2} + z\mathbf{e_3}$$

where $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ denote a right handed system of unit vectors; the coordinates of P in this sytem are (x, y, z). Sometimes we shall use the notation (x_1, x_2, x_3) instead of (x, y, z).

8.2 Laplace's equation

First, let us see how Laplace's[1] equation arises in the study of fluid mechanics.

Consider a fluid moving in a region of 3-dimensional space. Let $\mathbf{q}(x, y, z, t)$, $\rho(x, y, z, t)$ denote the velocity and density, respectively, of the fluid element at the point (x, y, z) at time t. The *continuity equation* which expresses the law of conservation of mass may be derived as follows:

The mass of fluid in a fixed volume v is $\int_v \rho\, dv$ and the rate at which this mass is changing is given by $\frac{\partial}{\partial t}\int \rho\, dv = \int \frac{\partial \rho}{\partial t}\, dv$. Any change in the total mass of fluid inside v must be accounted for by fluid crossing S, the smooth closed surface which bounds v. We note that the mass of fluid crossing the surface element dS per unit time is $\rho\mathbf{q}.\mathbf{n}dS = \rho\mathbf{q}.\mathbf{dS}$,

[1]Pierre Simon, Marquis de Laplace, (1749-1827). French mathematician, physicist, and astronomer.

where $\mathbf{n}$ denotes the unit normal to S, drawn out from v. This is clear, since the tangential component of $\mathbf{q}$ does not contribute; the normal component is $\mathbf{q}.\mathbf{n}$ and in unit time the mass of fluid crossing dS is the mass inside a cylinder of length $\mathbf{q}.\mathbf{n}$ (in the sense of the motion) and cross sectional area dS. The total mass crossing S per unit time is therefore $\int_S \rho\mathbf{q}.\mathbf{dS}$ Conservation of mass demands that

$$\int \frac{\partial \rho}{\partial t}\, dv = -\int_S \rho\mathbf{q}.\mathbf{dS} = -\int_S \mathbf{j}.\mathbf{dS},$$

where the mass flux vector $\mathbf{j}$ is given by $\mathbf{j} = \rho\mathbf{q}$.

By Gauss's divergence theorem it follows that

$$\int_v (\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j})\, dv = 0.$$

This formula holds for an arbitrary fixed volume v and assuming that the integrand is continuous it follows that, at each point of the fluid,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0 \tag{8.1}$$

This equation is called the *continuity equation* and in this context expresses the law of conservation of mass. If the fluid is incompressible we may treat ρ as a constant and the continuity equation reduces to

$$\operatorname{div}\mathbf{j} = 0. \tag{8.2}$$

If we assume additionally that the motion is irrotational, then $\operatorname{curl}\mathbf{q} = \mathbf{0}$, and we may write $\mathbf{q} = -\boldsymbol{\nabla}\phi$, where ϕ is usually referred to as the *velocity potential*. Here $\boldsymbol{\nabla}$ is Hamilton's operator *nabla* ($\nu\alpha\beta\lambda\alpha$ is the word used in classical Greek for a harp) given by

$$\boldsymbol{\nabla} = \mathbf{e_1}\frac{\partial}{\partial x} + \mathbf{e_2}\frac{\partial}{\partial y} + \mathbf{e_3}\frac{\partial}{\partial z} \tag{8.3}$$

It follows that for an incompressible fluid executing an irrotational motion that $\operatorname{div}(\boldsymbol{\nabla}\phi) = 0$. In other words ϕ satisfies the famous Laplace equation

$$\nabla^2\phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{8.4}$$

∇^2 is often referred to as Laplace's operator (in the coordinates (x, y, z)) or just the Laplacian. Explicitly

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{8.5}$$

In two space dimensions Laplace's equation becomes

$$\nabla^2\phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{8.6}$$

and in one space dimension just

$$\nabla^2\phi \equiv \frac{d^2\phi}{dx^2} = 0 \tag{8.7}$$

We can relate the theory of the two-dimensional motion of an incompressible fluid to the theory of analytic functions of a complex variable $\zeta = x + iy$ as follows. For such a motion we have $\mathbf{q}(x, y, z) = (u(x, y), v(x, y), 0)$. This means that the motion is always perpendicular to the z-axis and is independent of the z coordinate. The condition that the motion be irrotational may be expressed as

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}$$

and the solenoidal condition div $\mathbf{q} = 0$ may be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

It is proved in books on calculus that the second of these equations is precisely the condition for $-u\,dy + v\,dx$ to be the differential of a function ψ. so that

$$d\psi = -u\,dy + v\,dx, \quad u = -\frac{\partial\psi}{\partial y}, \; v = \frac{\partial\psi}{\partial x}$$

ψ is referred to as the *stream function* of the motion. Combining our equations for u, v in terms of ϕ, ψ we obtain

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{8.8}$$

which are precisely the **Cauchy-Riemann** equations relating the functions ϕ, ψ of the variables (x, y). Assuming that the first order derivatives of ϕ, ψ are continuous the Cauchy-Riemann equations guarantee that $\phi + i\psi$ is an analytic function of the complex variable $\zeta = x + iy$. ϕ and ψ both satisfy the two-dimensional Laplace equation.

The physical significance of the function ψ may be seen as follows: The path of a fluid element, known as a streamline, is given by the family of curves Γ with the property that the tangent at each point (x, y) of Γ is parallel to the velocity vector $\mathbf{q}(x, y)$. It is clear that the differential equation of this family of curves is

$$\frac{dy}{dx} = \frac{v(x, y)}{u(x, y)}.$$

Writing u and v interms of ψ this becomes

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0, \quad \frac{d}{dx}\psi(x, y(x)) = 0$$

using the chain rule of partial differentiation. It follows that $\psi(x, y) = C$, where C is a parameter; the streamlines are determined by the equation $\psi(x, y) = C$, where C is a parameter. Each streamline will be labelled by a different value of C.

Conversely, any analytic function $\omega(\zeta)$, $\zeta = x + iy$ describes two-dimensional irrotational flow of an incompressible fluid. We can see this as follows. Write $\omega(\zeta) = \phi(x, y) + i\psi(x, y)$ and define the velocity $\mathbf{q}$ at (x, y) by $\mathbf{q} = -\boldsymbol{\nabla}\phi$. Then $\operatorname{div}\mathbf{q} = 0$, since ϕ satisfies Laplace's equation (being the real part of an analytic function). Moreover, $\operatorname{curl}\mathbf{q} = \mathbf{0}$. It follows that $\mathbf{q}$ describes the irrotational flow of an incompressible fluid. The streamlines of the flow determined by $\mathbf{q}$ are given by the equation $\psi(x, y) = C$, where C is a real parameter, as described above.

Laplace's equation also arises in the theory of electricity. This time, suppose that v is an arbitrary volume in a conducting material in which electric charge is free to flow. Following the above notation we use $\mathbf{q}(x, y, z, t)$ and $\rho(x, y, z, t)$ to denote the velocity of the charge and the electric charge density, respectively, at the point (x, y, z) at time t. If we require *electric charge* to be conserved we obtain, exactly as above, the continuity equation expressing conservation of electric charge:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\mathbf{j} = 0, \quad \mathbf{j} = \rho\mathbf{q}.$$

In the case of steady currents $\mathbf{q}$ and ρ don't depend on t so the continuity equation reduces to

$$\operatorname{div}\mathbf{j} = 0.$$

It is found that for many conductors the vector $\mathbf{j}$ is proportional to the electric field $\mathbf{E}$. (If you're not familiar with this concept you can think of the value of $\mathbf{E}$ at a point (x, y, z) as being the *force* exerted on a unit charge placed at that point). Assuming Ohm's Law, we may write $\mathbf{j} = \sigma\mathbf{E}$, where σ is a constant called the electrical conductivity of the material. For steady currents the electric field satisfies $\operatorname{curl}\mathbf{E} = \mathbf{0}$ so that we may write $\mathbf{E} = -\boldsymbol{\nabla}\phi$, where ϕ is the *electric potential* . We conclude that for steady currents flowing in a material in which Ohm's Law holds, $\operatorname{div}(\sigma\boldsymbol{\nabla}\phi) = 0$. Assuming that σ is constant it follows that the electric potential satisfies Laplace's equation

$$\nabla^2\phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

The solution of problems in the theory of steady currents therefore reduces to finding appropriate solutions of Laplace's equation. In practice we require our solutions to satisfy so called *boundary conditions*. For example, we would require ϕ to assume a constant value on an electrode, and on a surface across which no charge flows we would demand that $\frac{\partial \phi}{\partial n} = 0$; the latter condition expresses the fact that on a surface across which no charge flows we must have $\mathbf{q}.\mathbf{n} = 0$ and since $\mathbf{q} = -\boldsymbol{\nabla}\phi$ this becomes $\frac{\partial \phi}{\partial n} = 0$. Recall (CM112A) that $\frac{\partial \phi}{\partial n}$ is the *directional derivative* of ϕ with respect to $\mathbf{n}$ i.e. the rate of change of ϕ with respect to distance measured parallel to $\mathbf{n}$. Boundary conditions are of fundamental importance in solving PDE s.

Another very important partial differential equation is the diffusion or heat equation; this equation arises in many braches of pure and applied mathematics; for example, in the theory of heat conduction and in probability theory.

8.3 The diffusion equation

Consider a fixed volume v (bounded by a smooth closed surface S) in some material which permits the flow of heat energy. Suppose that ρ represents the (mass) density of the material. The heat energy required to raise the temperature of a mass $\rho\, dv$ from temperature θ_0 to a temperature θ is $\rho\, dv(\theta - \theta_0)s$, where s is called the specific heat of the material. The heat energy in the volume v may therefore be regarded as $\int \rho(\theta - \theta_0)s\, dv$ and the rate at which this is changing with respect to time t is given by

$$\frac{\partial}{\partial t}\int \rho(\theta - \theta_0)s\, dv = \int \rho s \frac{\partial \theta}{\partial t}\, dv \tag{8.9}$$

(We've assumed that ρ doesn't depend on t which will usually be entirely reasonable.)

Bearing in mind the arguments applied above in the derivation of Laplace's equation we postulate a heat energy flow vector $\mathbf{j}$ so that $\int_\Sigma \mathbf{j}.\mathbf{d\Sigma}$ represents the amount of heat energy crossing a surface Σ per unit time. The principle of conservation of energy now requires that

$$\int_v \rho s \frac{\partial \theta}{\partial t}\, dv = -\int_S \mathbf{j}.\mathbf{dS}, \quad \int_v (\rho s \frac{\partial \theta}{\partial t} + \operatorname{div} \mathbf{j})\, dv = 0 \tag{8.10}$$

by an application of Gauss's theorem to transform the surface integral to a volume integral. This argument applies to an arbitrary volume v in the material and, assuming that the integrand is continuous, we conclude that at each point of the (heat) conducting material

$$\rho s \frac{\partial \theta}{\partial t} + \operatorname{div} \mathbf{j} = 0 \tag{8.11}$$

In order to make further progress we need to relate $\mathbf{j}$ to the temperature θ. What would be a reasonable assumption? First, $\mathbf{j}$ is a vector. Given a scalar function θ the first vector function which springs to mind is $\boldsymbol{\nabla}\theta$. We know that heat flows form regions of higher temperature to regions of lower temperature. It is therefore reasonable to write, on the basis of our physical and mathematical intuition,

$$\mathbf{j} = -k\boldsymbol{\nabla}\theta \tag{8.12}$$

where k is a (positive) constant depending on the material. Whether our guess is sound will depend on whether the resulting model leads to predictions in agreement with experiment. Anyway, with this assumption we obtain

$$\rho s \frac{\partial \theta}{\partial t} = \operatorname{div}(k\boldsymbol{\nabla}\theta) = k\nabla^2\theta.$$

Equivalently

$$\nabla^2\theta \equiv \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{\kappa}\frac{\partial \theta}{\partial t} \tag{8.13}$$

where κ is usually referred to as the *thermal conductivity* of the material. This important equation is called the *heat* or *diffusion* equation. In two and one space dimensions respectively it reads

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{1}{\kappa}\frac{\partial \theta}{\partial t} \tag{8.14}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial \theta}{\partial t} \tag{8.15}$$

One could write down various generalizations in higher dimensions which are of interest to pure mathematicians.

The heat or diffusion equation arises in many branches of mathematics, for example in probability theory.

The observation that, when suspended in water small pollen grains are found to be in a very animated and irregular state of motion was first investigated by Robert Brown, an English botanist, in 1827; as a result the phenomenon took on the name Brownian motion. The first convincing explanation of Brownian motion was given by Einstein[2] in 1905, the year in which he published the Special Theory of Relativity; 1905 was also the year in which he explained the photoelectric effect in terms of Planck's quantum theory!

Einstein assumed that the Brownian motion is caused by frequent impacts on the pollen grains by molecules of the liquid in which it is supended and that the motion of these molecules is so complicated that its effect on the pollen grain can only be decribed probabilistically in terms of frequent statistically independent impacts.

In Einstein's model he assumes that the impacts happen at times $0, \tau, 2\tau, 3\tau, \ldots, \tau$ being very small. For simplicity, consider the case of Brownian motion in one dimension – along the x–axis. It is assumed that when an impact occurs the probability that the pollen particle receives a displacement between Δ and $\Delta + d\,\Delta$is $\phi(\Delta) d\,\Delta$, so that $\int_{-\infty}^{\infty} \phi(\Delta)\, d\,\Delta = 1$. It is natural to assume that the probability density ϕ is even i.e. $\phi(-\Delta) = \phi(\Delta)$, from which it follows that $\int_{-\infty}^{\infty} \Delta\, \phi(\Delta)\, d\,\Delta = 0$.

Now suppose that $f(x,t)dx$ is the probability that the particle (pollen grain) lies between x and $x + dx$ at time t, so that $\int_{-\infty}^{\infty} f(x,t)\, dx = 1$.

[2]Albert Einstein (1879-1955) Professor of Theoretical Physics Berne (1909), Prague (1911), Berlin (1913), Professor of Mathematics Princeton (1933)

The probability that the particle lies between x and $x+dx$ at time $t+\tau$ is

$$\begin{aligned} f(x,t+\tau)\,dx &= \sum_{\Delta} \text{Prob(particle at } x+\Delta, \text{at time } t \text{ and receives a kick } -\Delta \\ &\quad \text{in the time gap } \tau) \\ &= \int_{-\infty}^{\infty} f(x+\Delta,t)\,dx\,\phi(-\Delta)\,d\Delta = \int_{-\infty}^{\infty} f(x+\Delta,t)\,dx\,\phi(\Delta)\,d\Delta \end{aligned}$$

Since τ is assumed very small we can write this equation as

$$\begin{aligned} f(x,t)+\frac{\partial f}{\partial t}(x,t)\tau+\cdots &= \int_{-\infty}^{\infty}\Big(f(x,t)+\frac{\partial f}{\partial x}(x,t)\,\Delta+\frac{\partial^2 f}{\partial x^2}(x,t)\Delta^2/2+\cdots\Big)\phi(\Delta)\,d\Delta \\ &= f(x,t)+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x,t)\int_{-\infty}^{\infty}\Delta^2\phi(\Delta)\,d\Delta \end{aligned}$$

since $\int_{-\infty}^{\infty}\phi(\Delta)\,d\Delta=1$ and $\int_{-\infty}^{\infty}\Delta\,\phi(\Delta)\,d\Delta=0$.

We note that $\int_{-\infty}^{\infty}\Delta^2\phi(\Delta)\,d\Delta$ is just the variance of the random variable Δ, so we denote it by σ^2. We conclude that

$$\frac{\partial f}{\partial t}\tau=\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial^2 f}{\partial x^2}=\frac{1}{\kappa}\frac{\partial f}{\partial t}, \qquad \text{where } \kappa=\frac{\sigma^2}{2\tau}.$$

We see that the probability density $f(x,t)$ satisfies the diffusion equation. We note in passing that a fundamental solution of this equation is

$$f(x,t)=\frac{1}{\sqrt{4\pi\kappa t}}\mathrm{e}^{-x^2/(4\kappa t)}.$$

This result is most readily derived by Fourier transform methods (see Chapter 15 — where we demonstrate this explicitly) but the fact that it is a solution can also be verified by substitution. The multiplicative factor $\frac{1}{\sqrt{4\pi\kappa}}$ in this expression is chosen so that $f(x,t)$ is indeed a probability density. In fact

$$\int_{-\infty}^{\infty} f(x,t)\,dx=\frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty}\mathrm{e}^{-x^2/(4\kappa t)}\,dx=1.$$

8.4 The wave equation

Imagine a taut string, stretched to a tension T, which executes small transverse vibrations in a direction perpendicular to the x-axis. Suppose that $u(x,t)$ denotes the displacement of the string at the point x at time t. Consider the motion of the portion PQ of the string, where P is the point $(x,u(x,t))$ and Q is the point $(x+dx,u(x+dx,t))$. Let ρ denote the linear density of the string i.e. the mass of the string per unit length. We note that PQ has length ds, where $ds=\sqrt{1+(\frac{\partial u}{\partial x})^2}\,dx$. Applying Newton's Second Law in the transverse direction we obtain

$$\rho ds\frac{\partial^2 u}{\partial t^2}=T\sin(\psi(x+dx,t))-T\sin\psi(x,t),$$

where ψ is the angle which the tangent to the string makes with the positive x-axis; of course, ψ will depend on x as well as t. Approximating the right hand of this equation by Taylor's theorem, and neglecting higher powers of dx, we obtain

$$\rho ds \frac{\partial^2 u}{\partial t^2} = T \frac{\partial}{\partial x} \sin\psi(x,t)\, dx.$$

For small ψ, $\sin\psi \simeq \tan\psi = \frac{\partial u}{\partial x}$ and it follows that

$$\rho \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}.$$

Since ψ is assumed small $(\frac{\partial u}{\partial x})^2$ is negligible and we conclude that u satisfies the PDE

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T}{\rho}.$$

This PDE is the *wave equation* in one space dimension. The analogous equation in three space dimensions is

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad \nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

In subsequent chapters we shall study some of these equations in greater detail.

Chapter 9

Basic ideas

A differential equation involving partial derivatives is called a Partial Differential Equation (PDE); we have already noted several examples in the preceding chapter.

Definition 9.1 *The order of a PDE is the order of the highest partial derivative occurring in it.*

A famous PDE, which has attracted a lot of attention in recent years, is the Korteweg-de Vries equation:

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u = u(x,t) \tag{9.1}$$

is a *third order* PDE whereas

$$\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial t}\right)^2 = 1, \quad u = u(x,t) \tag{9.2}$$

is a *first order* PDE.

Definition 9.2 *A PDE is linear if the unknown function u and any partial derivatives of u occur to the first degree only, and no products of u and its partial derivatives or products of partial derivatives of u appear in the equation.*

The Korteweg de-Vries equation is non-linear due to the presence of the term $6u\frac{\partial u}{\partial x}$ and equation 9.2 is also non-linear due to the presence of $(\frac{\partial u}{\partial t})^2$. Laplace's equation, the wave equation, and the diffusion equation which we derived in **Chapter 2** are all linear PDE s.

Definition 9.3 *A PDE is said to be homogeneous if each term contains either the dependent variable or one of its derivatives.*

Laplace's equation

$$\nabla^2\phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

is a linear, second order, homogeneous PDE but Poisson's equation

$$\nabla^2\phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \rho(x, y, z) \tag{9.3}$$

is a linear second order PDE which is not homogeneous.

Linear homogeneous PDE s can be written in the form $Lu = 0$, where L is a linear differential operator i.e. its action is such that

$$L(\lambda f + \mu g) = \lambda L f + \mu L g$$

where λ, μ are constants and f,g are admissible functions. This condition is clearly satisfied by the Laplacian operator L given by

$$L = \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

We note the following important property of linear homogeneous equations:

Suppose that $u_1, u_2, u_2, \ldots, u_n$ are n functions which satisfy the linear homogeneous PDE $Lu = 0$. Then the linear combination $\sum_{i=1}^{n} \lambda_i u_i$, where the λ_i are arbitrary constants, is also a solution of the PDE $Lu = 0$. The proof is easy:

$$L(\sum \lambda_i u_i) = \sum \lambda_i (L u_i) = 0.$$

The first step follows from the linearity of L, the second from the fact that $Lu_i = 0$, for $i = 1, 2, \ldots, n$.

This result is often referred to as the *superposition principle* for linear homogeneous PDE s. Note that we have established the principle for *finite* linear combinations although we shall often apply the superposition principle in a cavalier way to infinite sums.

We note in passing that if u_1, u_2 are solutions of the linear inhomogeneous equation $Lu = f$ then the difference, $u_1 - u_2$, satifies the homogeneous equation $Lu = 0$. This is clear since $L(u_1 - u_2) = Lu_1 - Lu_2 = f - f = 0$, by the linearity of the operator L. In particular, the difference of any two solutions of Poisson's equation satisfies Laplace's equation.

It is interesting to make a comparison between ordinary differential equations (ODE s) and PDE s. Recall that ODE s can arise by elimination of arbitrary constants in the manner of the following example.

Example 9.1 *Consider the equation*

$$y = \frac{x - A}{x + A},$$

where A is a parameter. As A varies we generate a family of curves in the x-y plane and each of these curves satisfies an ordinary differential equation which can be determined as follows. We have

$$\frac{dy}{dx} = \frac{2A}{(x+A)^2}, \quad A = \frac{x(1-y)}{1+y}, \quad x + A = \frac{2x}{1+y}$$

and straight forward elimination of the parameter A gives the ODE

$$2x\frac{dy}{dx} = 1 - y^2.$$

If we demand that a solution of this differential equation satisfy admissible initial conditions of the form $y = y_0$ when $x = x_0$ the value of A is determined. We have picked out from the family of curves $y = \frac{x-A}{x+A}$ the particular curve which passes through the point (x_0, y_0).

We now show by a simple example how PDE s can arise by elimination of *arbitrary functions*. (When we use the term arbitrary functions we do not intend the term arbitrary to be taken literally; the functions in question have to be sufficiently smooth for the relevant differentiations to be valid.)

Example 9.2 *Let $u(x,y) = f(x-y)+g(x+y)$, where f, g are arbitrary $\mathbf{C}^2$ functions. (We use the term $\mathbf{C}^n$ to mean that the function in question has continuous partial derivatives up to and including the n th order.)*

First, note that for a given choice of functions f, g the points $(x, y, u(x,y))$, as (x, y) vary, determine a surface (as opposed to a curve) in three dimensional space; as the functions f, g are varied we obtain a family of surfaces in three dimensional space. We now show that each of these surfaces satisfies the same PDE, whatever the choice of f, g. This is easy since

$$\frac{\partial u}{\partial x} = f'(x-y) + g'(x+y), \quad \frac{\partial^2 u}{\partial x^2} = f''(x-y) + g''(x+y),$$

($'$ denotes differentiation with respect to the argument of the function)

$$\frac{\partial u}{\partial y} = -f'(x-y) + g'(x+y), \quad \frac{\partial^2 u}{\partial y^2} = f''(x-y) + g''(x+y).$$

We see at once that, for every choice of f, g, u satisfies the PDE

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2},$$

which is none other than the wave equation (see Chapter 12). We note in particular that

$$\ln\cos(x-y) + \mathrm{e}^{(x+y)^4}, \quad \cosh(x-y) + \sin^2(x-y),$$

and so on, are solutions of the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

When considering ODE s it is usually desirable to obtain the *general solution*, if this is possible; in the case of PDE s a so-called general solution may be of little value since we are usually seeking a solution which satisfies *boundary conditions* and it is seldom easy to adapt the arbitrary functions which occur in a 'general' solution so as to satisfy the boundary conditions. With ODE s the situation would usually be more straight forward since we would be trying to fit a finite number of constants so as to satisfy the relevant initial or boundary conditions.

In what follows we shall consider only *linear* PDE s, especially the PDE s of Mathematical Physics which we derived in **Chapter 8**. Non-linear PDE s are often extremely difficult to deal with, and even the relatively simple Korteweg de-Vries equation 9.1 is far from straightforward; an enormous amount of effort has been invested in its study by some of the world's best mathematicians.

Chapter 10

Euler's equation

We note that the two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and the wave equation in one space dimension

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sigma^2}\frac{\partial^2 \phi}{\partial t^2}$$

are both particular cases of the equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2} = 0 \tag{10.1}$$

where a, b, c are real constants; this equation is called Euler's[1] equation after the famous Swiss mathematician.

Before showing how to find the general solution of Euler's equation we first note that two particular cases of Euler's equation are easy to solve.

1. Consider the equation

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{10.2}$$

Integration with respect to x gives $\frac{\partial u}{\partial x} = f_1(y)$, where f_1 is an arbitrary function of y. A further integration with respect to x then gives

$$u(x, y) = x f_1(y) + f_2(y),$$

where f_2 is also an arbitrary function of y, as the general solution. Note that the general solution involves two arbitrary functions rather than two arbitrary constants.

[1]Leonhard Euler (1707-1783) Swiss mathematician noted for his work in analysis, particularly the Calculus of Variations. He was the first to write down the formula $e^{i\pi} = -1$.

2. Consider the equation

$$\frac{\partial^2 u}{\partial x\,\partial y} = 0 \tag{10.3}$$

Integration with respect to x gives $\frac{\partial u}{\partial y} = g(y)$, where g is an arbitrary function. Integration with respect to y now gives

$$u(x,y) = \int g(y)\,dy + g_2(x),$$

where g_2 is an arbitrary function of x. We can tidy up the result by writing

$$u(x,y) = g_1(y) + g_2(x),$$

where $g_1(y) = \int g(y)\,dy$. Again, the general solution involves two arbitrary functions rather than two arbitrary constants.

10.1 Euler's equation

We now show by a suitable change of variable that we can reduce Euler's equation 10.1 to an equation of type **1** or **2**. Euler's equation is a linear PDE with constant coefficients so it is entirely natural to consider a linear change of variable $(x,y) \mapsto (\xi,\eta)$ where

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \zeta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \xi = \alpha x + \beta y, \quad \eta = \gamma x + \zeta y.$$

It is important that the transformation has an inverse, so that given (ξ,η) we can solve uniquely for (x,y); this demands that the matrix of the transformation be non-singular — so that $\alpha\zeta - \beta\gamma \neq 0$.

Application of the chain rule of partial differentiation gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

which immediately gives

$$\frac{\partial u}{\partial x} = \alpha\frac{\partial u}{\partial \xi} + \gamma\frac{\partial u}{\partial \eta}, \text{ so that } \frac{\partial}{\partial x} \equiv \alpha\frac{\partial}{\partial \xi} + \gamma\frac{\partial}{\partial \eta}$$

and

$$\frac{\partial u}{\partial y} = \beta\frac{\partial u}{\partial \xi} + \zeta\frac{\partial u}{\partial \eta}, \text{ so that } \frac{\partial}{\partial y} \equiv \beta\frac{\partial}{\partial \xi} + \zeta\frac{\partial}{\partial \eta}$$

We then have

$$\frac{\partial^2 u}{\partial x^2} = \left(\alpha\frac{\partial}{\partial \xi} + \gamma\frac{\partial}{\partial \eta}\right)\left(\alpha\frac{\partial u}{\partial \xi} + \gamma\frac{\partial u}{\partial \eta}\right) = \alpha^2\frac{\partial^2 u}{\partial \xi^2} + 2\alpha\gamma\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \gamma^2\frac{\partial^2 u}{\partial \eta^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \Big(\beta\frac{\partial}{\partial \xi} + \zeta\frac{\partial}{\partial \eta}\Big)\Big(\beta\frac{\partial u}{\partial \xi} + \zeta\frac{\partial u}{\partial \eta}\Big) = \beta^2\frac{\partial^2 u}{\partial \xi^2} + 2\beta\zeta\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \zeta^2\frac{\partial^2 u}{\partial \eta^2},$$

and

$$\frac{\partial^2 u}{\partial x\,\partial y} = \Big(\alpha\frac{\partial}{\partial \xi} + \gamma\frac{\partial}{\partial \eta}\Big)\Big(\beta\frac{\partial u}{\partial \xi} + \zeta\frac{\partial u}{\partial \eta}\Big) = \alpha\beta\frac{\partial^2 u}{\partial \xi^2} + (\alpha\zeta + \gamma\beta)\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \gamma\zeta\frac{\partial^2 u}{\partial \eta^2}.$$

Substituting these expressions into Euler's equation and gathering terms gives

$$\frac{\partial^2 u}{\partial \xi^2}(a\alpha^2 + 2b\alpha\beta + c\beta^2) + \frac{\partial^2 u}{\partial \eta^2}(a\gamma^2 + 2b\gamma\zeta + c\zeta^2) + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta}(a\alpha\gamma + b(\alpha\zeta + \gamma\beta) + c\beta\zeta) = 0.$$

This suggests that we choose $\alpha = 1,\ \gamma = 1$ and $\beta = \lambda_1,\ \zeta = \lambda_2$, where $\lambda_1,\ \lambda_2$ are roots of the quadratic equation

$$a + 2b\lambda + c\lambda^2 = 0.$$

With this choice the determinant of our transformation $(x, y) \mapsto (\xi, \eta)$ is equal to $\lambda_2 - \lambda_1$ and the requirement that the transformation be non-singular demands that the quadratic equation has *distinct* roots. Asssuming that this is the case for the moment, we see that Euler's equation reduces to

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta}(a + b(\lambda_1 + \lambda_2) + c\lambda_1\lambda_2) = 0.$$

Now,

$$\lambda_1 + \lambda_2 = -\frac{2b}{c}, \quad \lambda_1\lambda_2 = \frac{a}{c}$$

and we find that

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta}\Big(2\frac{(ac - b^2)}{c}\Big) = 0.$$

The term in brackets is non-zero since the roots $\lambda_1,\ \lambda_2$ have been assumed to be distinct and it follows that Euler's equation becomes

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0.$$

This equation has general solution (see the argument above)

$$u = f(\xi) + g(\eta) = f(x + \lambda_1 y) + g(x + \lambda_2 y),$$

where f, g are arbitrary $\mathbf{C}^2$ functions.

What happens if the quadratic equation

$$a + 2b\lambda + c\lambda^2 = 0$$

has equal roots? If the roots are equal we can't apply the above reasoning since the transformation $(x, y) \mapsto (\xi, \eta)$ is singular i.e. it doesn't have an inverse; the existence of the inverse was crucial for the application of the Chain Rule. However, we can modify our argument slightly, as follows.

We write $\lambda_1 = \lambda_2 = \lambda = -\frac{b}{c} = -\frac{a}{b}$ (since $b^2 - ac = 0$ in this case.) An examination of our earlier argument suggests that we make the transformation $(x, y) \mapsto (\xi, \eta)$ where

$$\xi = x + \lambda y,\ \eta = \gamma x + \zeta y,$$

and γ and ζ are such that $\zeta \neq \lambda\gamma$ (we impose this constraint in order to ensure that our transformation is non-singular) but are otherwise arbitrary. This means that we've chosen $\alpha = 1,\ \beta = \lambda$ in the notation used above.

With this choice the coefficient of $\frac{\partial^2 u}{\partial \xi^2}$ is obviously zero whilst the coeffiecient of $\frac{\partial^2 u}{\partial \xi\, \partial \eta}$ is

$$a\alpha\gamma + b(\alpha\zeta + \gamma\beta) + c\beta\zeta = a\gamma + b(\zeta + \gamma\ \lambda) + c\lambda\zeta = \gamma(a + b\lambda) + \zeta(b + c\lambda) = 0.$$

The coefficient of $\frac{\partial^2 u}{\partial \eta^2}$ is

$$a\gamma^2 + 2b\gamma\zeta + c\zeta^2.$$

This cannot be zero since if it were we would have

$$a + 2b(\zeta/\gamma) + c(\zeta/\gamma)^2 = 0$$

and this would imply that $\zeta/\gamma = \lambda$, which we ruled out above. It follows that in this case Euler's equation reduces to

$$\frac{\partial^2 u}{\partial \eta^2} = 0$$

which, as we noted above, has general solution

$$u = \eta f(\xi) + g(\xi) = (\gamma x + \zeta y) f(x + \lambda y) + g(x + \lambda y),$$

where f, g are arbitrary $\mathbf{C}^2$ functions and γ and ζ are any numbers such that $\gamma \neq \lambda\zeta$.

We can classify Euler's equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\, \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0$$

according to the solutions of the quadratic equation

$$a + 2b\lambda + c\lambda^2 = 0.$$

This quadratic is sometimes referred to as the *auxiliary* equation and, as noted above, enables us to write down the general solution of Euler's equation. If the roots of the auxiliary equation are real and distinct we say that Euler's equation is **hyperbolic**, if the roots are coincident we say that it is **parabolic**, whilst if the roots are conjugate complex numbers we say that it is **elliptic**.

10.2 Examples

Example 10.1 *The wave equation in one space dimension is*

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

The auxiliary equation is

$$1 - \frac{\lambda^2}{c^2} = 0, \quad \lambda = \pm c.$$

Since the roots are real the wave equation is hyperbolic and has general solution

$$u(x,t) = f_1(x - ct) + f_2(x + ct),$$

where f_1, f_2 *are arbitrary functions. This solution is due to D'Alembert*[2]*.*

We shall study the wave equation in more detail in a later chapter.

Example 10.2 *The two-dimensional Laplace equation is*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The auxiliary equation is

$$1 + \lambda^2 = 0, \quad \lambda = \pm i.$$

Since the roots are complex Laplace's equation is elliptic and has general solution

$$u(x,t) = g_1(x + iy) + g_2(x - iy),$$

where g_1, g_2 *are arbitrary functions.*

We recall that the real and imaginary parts of any analytic function $g(z), \quad z = x + iy$ satisfies the two-dimensional Laplace equation (see Chapter 3).

[2] Jean Le Rond d'Alembert (1717-1783), French mathematician and encylopedist

Chapter 11

Symmetry and PDE s

Given a PDE it is interesting to consider its symmetry groups; these are groups of transformations which leave the PDE *invariant*. By way of illustration we look at two examples: Laplace's equation in three dimensions and the wave equation in one space dimension.

11.1 Laplace's equation

Consider Laplace's equation in three space dimensions:

$$\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} = 0.$$

For reasons which will soon become apparent we have adopted a subscript notation, (x_1, x_2, x_3) rather than (x, y, z).

Suppose that a point with coordinates (x_1, x_2, x_3) has coordinates $(x_1' x_2', x_3')$ with respect to axes related to the original axes by a rotation. Rotations preserve lengths so, writing

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix},$$

we require $x^T x = x'^T x'$, with $x' = \Lambda x$, where Λ is the 3×3 matrix describing the rotation. This condition implies that Λ must satisfy $x^T x = x^T \Lambda^T \Lambda x$, $\forall x \in \mathbf{R}^3$. It is not very hard to deduce that $\Lambda^T \Lambda = \Lambda \Lambda^T = I$, where I is the 3×3 unit matrix. In other words, the rotation matrix Λ has to be *orthogonal*. It follows that

$$x' = \Lambda x \Rightarrow x = \Lambda^{-1} x' = \Lambda^T x'$$

so that

$$x_j' = \sum_{k=1}^{3} \Lambda_{jk} x_k, \quad x_j = \sum_{k=1}^{3} \Lambda_{jk}^T x_k' = \sum_{k=1}^{3} \Lambda_{kj} x_k'.$$

It now follows from the chain rule that

$$\frac{\partial u}{\partial x_i} = \sum_j \frac{\partial u}{\partial x'_j}\frac{\partial x'_j}{\partial x_i} = \sum_j \Lambda_{ji}\frac{\partial u}{\partial x'_j}.$$

Hence

$$\begin{aligned}
\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} &= \sum_i \frac{\partial^2 u}{\partial {x_i}^2} = \sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) \\
&= \sum_i\sum_j\sum_k \Lambda_{ji}\frac{\partial^2 u}{\partial x'_j\,\partial x'_k}\frac{\partial x'_k}{\partial x_i} = \sum_{i,j,k}\Lambda_{ji}\Lambda_{ki}\frac{\partial^2 u}{\partial x'_j\,\partial x'_k} \\
&= \sum_{j,k}(\Lambda\Lambda^T)_{kj}\frac{\partial^2 u}{\partial x'_j\,\partial x'_k} = \sum_{j,k}\delta_{k,j}\frac{\partial^2 u}{\partial x'_j\,\partial x'_k} = \sum_j \frac{\partial^2 u}{\partial {x'_j}^2}
\end{aligned}$$

In other words

$$\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} = \frac{\partial^2 u}{\partial {x'_1}^2} + \frac{\partial^2 u}{\partial {x'_2}^2} + \frac{\partial^2 u}{\partial {x'_3}^2}.$$

If

$$\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} = 0$$

then

$$\frac{\partial^2 u}{\partial {x'_1}^2} + \frac{\partial^2 u}{\partial {x'_2}^2} + \frac{\partial^2 u}{\partial {x'_3}^2} = 0.$$

This means that Laplace's equation is invariant with respect to rotations; it takes the same mathematical form in two coordinate systems which are related by a rotation.

This invariance with respect to the group of rotations is perhaps one of the reasons why the Laplacian occurs so frequently in the equations of mathematical physics; if we believe that there is no preferred direction in space then the laws of nature should be invariant with respect to rotations.

It's more or less obvious that Laplace's equation is invariant with repect to the group of translations in 3-dimensional space give by

$$(x_1, x_2, x_3) \mapsto (x'_1, x'_2, x'_3), \quad x'_i = x_i + a_i \ (i = 1, 2, 3).$$

11.2 The wave equation

Now consider the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

This equation is linear and we therefore consider the possibility that it is invariant with respect to linear transformations of the type $(x,t) \mapsto (x',t')$, where

$$x' = \alpha x + \beta t, \quad t' = \gamma x + \zeta t.$$

We note that

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial t} = \beta\frac{\partial u}{\partial x'} + \zeta\frac{\partial u}{\partial t'}$$

so that

$$\frac{\partial}{\partial t} = \beta\frac{\partial}{\partial x'} + \zeta\frac{\partial}{\partial t'}$$

Similarly

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial t'}\frac{\partial t'}{\partial x}, = \alpha\frac{\partial u}{\partial x'} + \gamma\frac{\partial u}{\partial t'}$$

so that

$$\frac{\partial}{\partial x} = \alpha\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'}$$

Using these operators to tansform the wave equation to the new coordinates we obtain

$$\left(\alpha^2\frac{\partial^2 u}{\partial x'^2} + 2\alpha\gamma\frac{\partial^2 u}{\partial x'\,\partial t'} + \gamma^2\frac{\partial^2 u}{\partial t'^2}\right) = \frac{1}{c^2}\left(\beta^2\frac{\partial^2 u}{\partial x'^2} + 2\beta\zeta\frac{\partial^2 u}{\partial x'\,\partial t'} + \zeta^2\frac{\partial^2 u}{\partial t'^2}\right).$$

It follows that the wave equation is invariant with respect to the transformation considered if

$$\alpha\gamma = \frac{\beta\zeta}{c^2}, \quad \text{and} \quad \left(\alpha^2 - \frac{\beta^2}{c^2}\right) = \left(\frac{\zeta^2}{c^2} - \gamma^2\right)c^2 \tag{11.1}$$

We note in passing that the wave equation is not invariant with respect to the Galilean transformation of Newtonian Mechanics:

$$(x,t) \mapsto (x',t'), \quad x' = x - Vt, \ t' = t$$

for in this case $\alpha = 1, \ \gamma = 0, \ \beta = -V, \ \zeta = 1$ and the first of equations 11.1 cannot be satisfied. We can satisfy equations 11.1 by writing

$$\alpha = \cosh\sigma, \ \frac{\beta}{c} = -\sinh\sigma, \ \sigma \in \mathbf{R} \tag{11.2}$$

Substitution into the second of equations 11.1 gives

$$\zeta^2 - \gamma^2 c^2 = 1 \text{ since } \cosh^2\sigma - \sinh^2\sigma = 1 \tag{11.3}$$

and into the first of equations 11.1 gives

$$\gamma\cosh\sigma = -\frac{\zeta\sinh\sigma}{c}, \ \gamma = -\frac{\zeta\tanh\sigma}{c} \tag{11.4}$$

It follows from equations 11.4 and 11.3 that

$$\zeta^2(1 - \tanh^2\sigma) = 1.$$

This equation can be satisfied by the choice $\zeta = \cosh\sigma$ and we deduce from equation 11.4 that

$$\gamma = -\frac{\sinh\sigma}{c} \tag{11.5}$$

It follows that the wave equation is invariant with respect to the transformation

$$(x,t) \mapsto (x',t'), \quad \text{where } \begin{pmatrix} x' \\ t' \end{pmatrix} = M(\sigma)\begin{pmatrix} x \\ t \end{pmatrix},$$

where

$$M(\sigma) = \begin{pmatrix} \cosh\sigma & -c\sinh\sigma \\ -\frac{\sinh\sigma}{c} & \cosh\sigma \end{pmatrix}, \quad \sigma \in \mathbf{R} \tag{11.6}$$

Note that for all σ $\det M(\sigma) = 1$.

It is easy to check that the set of matrices $L = \{M(\sigma),\ \sigma \in \mathbf{R}\}$ forms a group under matrix multiplication:

First,

$$\begin{aligned} M(\sigma_1)M(\sigma_2) &= \begin{pmatrix} \cosh\sigma_1 & -c\sinh\sigma_1 \\ -\frac{\sinh\sigma_1}{c} & \cosh\sigma_1 \end{pmatrix}\begin{pmatrix} \cosh\sigma_2 & -c\sinh\sigma_2 \\ -\frac{\sinh\sigma_2}{c} & \cosh\sigma_2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\sigma_1\cosh\sigma_2 + \sinh\sigma_1\sinh\sigma_2 & -c\cosh\sigma_1\sinh\sigma_2 - c\sinh\sigma_1\cosh\sigma_2 \\ -\frac{\sinh\sigma_1\cosh\sigma_2 + \cosh\sigma_1\sinh\sigma_2}{c} & \cosh\sigma_1\cosh\sigma_2 + \sinh\sigma_1\sinh\sigma_2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\sigma_1+\sigma_2) & -c\sinh(\sigma_1+\sigma_2) \\ -\frac{\sinh(\sigma_1+\sigma_2)}{c} & \cosh(\sigma_1+\sigma_2) \end{pmatrix} \\ &= M(\sigma_1+\sigma_2) \end{aligned}$$

This formula establishes that L is closed under matrix multiplication. The group identity is the matrix $M(\sigma = 0) = I$, and the group inverse of $M(\sigma)$ is $M(-\sigma)$.

If we parametrise σ in terms of a new real parameter V by

$$\cosh\sigma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}, \quad \sinh\sigma = \frac{\frac{V}{c}}{\sqrt{1-\frac{V^2}{c^2}}}, \quad |V| < c,$$

(so that $\cosh^2\sigma - \sinh^2\sigma = 1$) we see from equation 11.6 that the wave equation is invariant under the group of transformations $(x,t) \mapsto (x',t')$, where

$$x' = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}}, \quad t' = \frac{t - Vx/c^2}{\sqrt{1-\frac{V^2}{c^2}}}.$$

These are the Lorentz transformations, well known to students of the Special Theory of Relativity.

Chapter 12

The wave equation

12.1 Introduction

In this chapter we consider the wave equation in one space dimension:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

In Chapter 10 we obtained the general solution

$$u(x,t) = f(x - ct) + g(x + ct) \tag{12.1}$$

A physical interpretation of this solution can be obtained as follows. Imagine a wave which moves along the positive x-axis with constant speed c and *without change of shape*. Suppose that at time $t = 0$ the shape of the wave is given by $u = f(x)$. An observer O, who moves along the positive x-axis with speed c, will note that the shape of the wave has not changed (relative to O), and will therefore use the equation $u = f(X)$, where $X = x - ct$ (see diagram) to describe it. It follows that $u = f(x - ct)$ describes a wave moving along the positive x-axis, with speed c, *without change of shape*. Similarly, $u = g(x + ct)$ describes a wave travelling along the negative x-axis with speed c and without change of shape. D'Alembert's solution, equation 12.1, is a superposition of two waves, one moving along the positive x-axis with speed c, the other moving along the negative x-axis with speed c.

We can picture the solutions of the wave equation geometrically as follows. Each choice of the functions f, g defines a solution. For each point (x, t) we can plot the point $(x, t, u(x, t))$, with $u(x, t) = f(x - ct) + g(c + ct)$, in three dimensional space. As (x, t) vary we generate a *surface* in three dimesional space, a solution surface of the wave equation. As we alter f, g we generate a family of such surfaces. In practical applications we often require to find a solution satisfying certain initial conditions or boundary conditions; geometrically this amounts to selecting the surface (or surfaces) which satisfy these conditions.

12.2 The infinite string

As noted previously, general solutions of PDE s are not always useful, especially when we are trying to find a solution satisfying specified boundary conditions. However, D'Alembert's solution can be used to solve the following problem.

Example 12.1 *Consider an infinite string and suppose that we impose Cauchy conditions:*

$$u(x, 0) = \phi(x), \quad \frac{\partial u}{\partial t}(x, 0) = \psi(x), \quad x \in \mathbf{R} \tag{12.2}$$

The first of these conditions specifies the initial shape of the string and the second gives the initial velocity of each point x on the string. We require to find the solution $u(x, t)$ of the wave equation, for all $x \in \mathbf{R}$, $\forall t \geq 0$, which satisfies the initial conditions 12.2. Another, less physical way of looking at the initial conditions is as follows: we require to find the solution of the wave equation in the upper-half plane $t \geq 0$, given the value of u and its normal derivative on the line $t = 0$.

We impose the initial conditions on equation 12.1. This requires

$$\phi(x) = f(x) + g(x) \tag{12.3}$$

$$\psi(x) = -cf'(x) + cg'(x) \tag{12.4}$$

Integrating equation 12.4 we obtain

$$f(x) - g(x) = -\frac{1}{c}\int_{x_0}^{x} \psi(z)\, dz + k \tag{12.5}$$

where $k = f(x_0) - g(x_0)$ *is constant. Adding and subtracting equations 12.3 and 12.5 immediately gives*

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_{x_0}^{x} \psi(z)\,dz + \frac{k}{2} \tag{12.6}$$

$$g(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_{x_0}^{x} \psi(z)\,dz - \frac{k}{2} \tag{12.7}$$

Substituting for f, g *in D'Alembert's solution we obtain the solution to our problem:*

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) - \frac{1}{2c}\int_{x_0}^{x-ct} \psi(z)\,dz + \frac{1}{2c}\int_{x_0}^{x+ct} \psi(z)\,dz \\ &= \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x_0} \psi(z)\,dz + \frac{1}{2c}\int_{x_0}^{x+ct} \psi(z)\,dz \\ &= \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(z)\,dz \end{aligned} \tag{12.8}$$

In particular, if the string is released from rest at time $t = 0$ (so that $\psi(x) = 0$) the solution is

$$u(x,t) = \frac{1}{2}(\phi(x-ct)+\phi(x+ct))$$

If the initial shape is given by the Gaussian $\phi(x) = \mathrm{e}^{-x^2}$ then

$$u(x,t) = \frac{1}{2}(\mathrm{e}^{-(x-ct)^2} + \mathrm{e}^{(x+ct)^2})$$

We can picture this as follows. We split the initial profile into two equal parts, each given by $\frac{1}{2}\mathrm{e}^{-x^2}$. One component then moves off with speed c along the positive x-axis without change of shape, the other with speed c along the negative x-axis and without change of shape; the resulting profile is a superposition of the two.

If $\phi(x) = 0, \psi(x) = \frac{1}{1+x^2}$ $\forall x \in \mathbf{R}$ (so that initially the string is along the x-axis and is set in motion with speed at the point x equal to $\psi(x)$) we see from equation 12.8 that

$$u(x,t) = \frac{1}{2c}(\arctan(x+ct) - \arctan(x-ct))$$

In the above example we considered an infinite string but we are frequently interested in the vibrations of a finite string fixed at both ends (e.g. the vibrations of a piano string). As a prelude to a discussion of such problems we first give a brief discussion of Fourier[1] series.

[1] Jean Baptiste Joseph Fourier (1768-1830) Noted for his work on the theory of heat in which he developed the method named after him.

12.3 Fourier Series

The classical theory of Fourier series deals with functions $f : [-\pi, \pi] \to \mathbf{R}$ which are periodic with period 2π, $f(x) = f(x + 2\pi)$, and investigates the circumstances in which it is possible to expand $f(x)$ in terms of functions in the set A given by

$$A = \{1, \cos x, \cos 2x, \cos 3x, \dots, \cos nx, \dots, \sin x, \sin 2x, \sin 3x, \dots, \sin nx, \dots\}$$

(all of which are periodic with period 2π) in the sense that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{12.9}$$

for some choice of the coefficients a_n, b_n.

Granted the possibility of such an expansion we may ask: What are the values of the coefficients a_n, b_n in relation to f? To this end we first note that the functions in A are *orthogonal* in the sense that if ϕ, $\psi \in A$, $\phi \neq \psi$, then

$$(\phi, \psi) = \int_{-\pi}^{\pi} \phi(x)\psi(x)\, dx = 0.$$

(We can think of (ϕ, ψ) as a *scalar product* of the functions ϕ, ψ) Explicitly,

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = \pi\delta_{n,m}, \qquad \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = \pi\delta_{n,m} \tag{12.10}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\, dx = 0, \qquad \int_{-\pi}^{\pi} (1)\sin(nx)\, dx = \int_{-\pi}^{\pi} (1)\cos(nx) = 0, \tag{12.11}$$

for every choice of the integers m, n; here $\delta_{n,m}$ is the Kronecker δ which takes the value 1 when m, n are equal and the value zero otherwise.

Referring to formula 12.9 we may now formally compute the coefficients a_n, b_n as follows: Integrate both sides with respect to x from $-\pi$ to π. to obtain

$$\frac{1}{2}a_0 \int_{-\pi}^{\pi} dx = \int_{-\pi}^{\pi} f(x)\, dx, \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \tag{12.12}$$

In this computation we have assumed that term by term integration of the series is valid.

Next, multiply both sides of equation 12.9 by $\cos(mx)$ and integrate from $-\pi$ to π. Formally, we obtain

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos mx\, dx &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx \\ &+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\cos(mx)\, dx = \sum_{n=1}^{\infty} a_n\delta_{n,m}\pi = \pi a_m,\end{aligned}$$

using the orthogonality relations 12.10, 12.11. This procedure therefore yields

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx, \qquad m = 1, 2, 3, \ldots \tag{12.13}$$

In a similar way, multiplying 12.9 by $\sin(mx)$ and integrating from $-\pi$ to π, we find that

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx, \qquad m = 1, 2, 3, \ldots \tag{12.14}$$

Given f, equations 12.12, 12.13, 12.14 determine the *Fourier coefficients* of f with respect to the orthogonal functions in the set A. For suitably well behaved functions f the Fourier coefficients exist and the resulting series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx), \tag{12.15}$$

(where $a_0,\ a_n,\ b_n$ are computed from equations 12.12, 12.13, 12.14) is called the *Fourier Series* associated with f.

All the above arguments were formal; they only suggest the *possibilty* that certain results *may* be true. For example, we implicitly assumed that the Fourier series converged to $f(x)$, and that we could integrate it term by term. The question of the convergence (or divergence) of the series 12.15 (the coefficients $a_n,\ b_n$ being determined by 12.12, 12.13, 12.14) is fundamental; also, if the series is convergent, does it converge to $f(x)$? The answer to such questions will obviously depend on what we assume about the function f and much of the classical theory is devoted to examining the convergence of the Fourier series under different sets of assumptions about f. A readable introduction, which might be intelligible to a student with an interest in analysis who had attended CM221A, is contained in the book *A First Course in Partial Differential Equations* by H Weinberger. (There are several copies of this book in the library, the most up to date version being the Dover edition published in 1995). Weinberger proves, for example, that the Fourier series 12.15 converges to $\frac{1}{2}(f(x+0)+f(x-0))$ subject to the assumptions

- f is absolutely integrable i.e. $\int_{-\pi}^{\pi}|f(x)|\,dx < \infty$
- f has a uniformly bounded derivative in some set $I(x,\epsilon) = (x-\epsilon, x)\cup(x, x+\epsilon)$, for some $\epsilon > 0$.

The second condition demands that f is differentiable in the set $I(x,\epsilon)$ (although it need not be differentiable at the point x itself) and that there is a positive constant K such that $|f'(z)| \le K \quad \forall z \in I$. The function $\sin(1/x)$, for example, would not satisfy this condition near the origin where its derivative oscillates wildly.

(Recall also that $f(x+0)$ is the limit of $f(z)$ as $z \to x$ from the right, and $f(x-0)$ is the limit of $f(z)$ as $z \to x$ from the left; if f is continuous at x then $f(x-0) = f(x+0) = f(x)$ and $\frac{1}{2}(f(x+0)+f(x-0)) = f(x)$.)

Perhaps it is worth mentioning that all proofs relating to the convergence of the Fourier series 12.15 start by showing that the sum $S_N(x)$ of the first N terms of the series 12.15 is given by (try to prove this)

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\frac{\sin(N+1/2)(x-u)}{\sin(x-u)/2}\,du.$$

This is the easy part; the hard part of the subsequent arguments involves discussion of the behaviour of $S_N(x)$ as $N \to \infty$. The arguments which are required and the degree of difficulty depends, of course, on what one chooses to assume about f.

If instead of assuming that our functions are defined on $[-\pi, \pi]$ and are periodic with period 2π we assume that they are defined on $[-l, l]$ and that they have period $2l$ all the above arguments hold with very minor modifications. In particular we have to deal with the orthogonal functions in the set

$$A_l = \{1, \cos(\pi x/l), \ldots, \cos(n\pi x/l), \ldots \sin(\pi x/l), \ldots, \sin(n\pi x/l), \ldots\}$$

all of which are periodic with period $2l$. They satisfy the orthogonality relations

$$\int_{-l}^{l} \cos(nx/l)\cos(mx/l)\,dx = l\delta_{n,m}, \qquad \int_{-l}^{l} \sin(nx/l)\sin(mx/l)\,dx = l\delta_{n,m} \tag{12.16}$$

$$\int_{-l}^{l} \sin(nx/l)\cos(mx/l)\,dx = 0, \qquad \int_{-l}^{l}(1)\sin(nx/l)\,dx = \int_{-l}^{l}(1)\cos(nx/l)\,dx = 0 \tag{12.17}$$

The corresponding Fourier series is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x/l) + \sum_{n=0}^{\infty} b_n\sin(n\pi x/l), \tag{12.18}$$

where the coeficients a_n, b_n are given by

$$a_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos(m\pi x/l)\,dx, \qquad m = 0, 1, 2, 3, \ldots, \tag{12.19}$$

$$b_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin(m\pi x/l)\,dx, \qquad m = 1, 2, 3, \ldots \tag{12.20}$$

We note in passing that if f is an *odd* function on $[-l, l]$, so that $f(-x) = -f(x)$, then

$$a_n = 0\ \forall n, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin(n\pi x/l)\,dx,$$

whilst if f is an *even* function

$$b_n = 0\ \forall n, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos(n\pi x/l)\,dx.$$

This is because the functions $\cos(n\pi x/l)$, $\sin(n\pi x/l)$ are even and odd functions respectively, and the fact that the product of two odd functions or two even functions is even, whilst the product of an even and an odd function is odd.

Consider the following example.

Example 12.2 *Find the Fourier series for* $\cosh ax$, $a \neq 0$, $-\pi \leq x \leq \pi$. *By putting* $x = 0$ *and then* $x = \pi$ *in turn obtain the formulae*

$$\frac{\pi}{\sinh a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty} (-1)^n \frac{2a}{a^2 + n^2},$$

$$\pi \coth a\pi = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 + n^2}.$$

Consider the function $f : [-\pi, \pi] \to \mathbf{R}$, $f(x) = \cosh ax$ *We can extend the definition of* f *to all real* x *by setting* $f(x + 2\pi) = f(x)$. *It is clear that the convergence criteria which we discussed above are satisfied and the Fourier series will converge to* $f(x) = \cosh ax$ *at all points* $x \in (-\pi, \pi)$. *A little refelction, in the light of earlier considerations, also shows that at the points* $\pm\pi$ *the Fourier series converges to* $\frac{1}{2}(f(-\pi + 0) + f(\pi - 0)) = \cosh \pi a$, *since* cosh *is an even continuous function.*

The Fourier series for f *is*

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where the Fourier coefficients a_n, b_n *are given by*

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cosh ax \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cosh ax \sin nx \, dx.$$

All the b_n *are zero because* $\cosh ax \sin nx$ *is an odd function. We note that*

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \cosh ax \, dx = \frac{2}{\pi a} \sinh \pi a,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cosh ax \cos nx \, dx = \frac{2}{\pi} (-1)^n \frac{\sinh \pi a}{a} - \frac{n^2}{a^2} a_n,$$

after integrating by parts twice. It follows that

$$a_n = \frac{(2a \sinh \pi a)(-1)^n}{\pi(a^2 + n^2)}.$$

Bearing in mind the above remarks concerning the convergence of the series we conclude that, for all $x \in [-\pi, \pi]$

$$\cosh ax = \frac{1}{\pi a} \sinh \pi a + \sum_{n=1}^{\infty} \frac{(2a \sinh \pi a)(-1)^n}{\pi(a^2+n^2)} \cos nx.$$

Putting $x = 0$ *in this formula immediately gives*

$$\frac{\pi}{\sinh \pi a} = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{(a^2+n^2)}(-1)^n$$

whilst setting $x = \pi$ *gives*

$$\pi \coth \pi a = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{(a^2+n^2)}.$$

Next we consider the vibrations of a string fixed at both ends.

12.4 Finite string

Consider the following problem.

Example 12.3 *Consider a finite string whose vibrations satisfy the wave equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

subject to the conditions:

$$u(x,0) = \phi(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in [0,l], \quad u(0,t) = 0, u(l,t) = 0 \ \ \forall t \geq 0 \tag{12.21}$$

The first condition specifies the shape of the string at $t = 0$*, the second says that the string starts from rest; the third and fourth conditions imply that the string is fixed at* $x = 0$ *and* $x = l$ *respectively.*

First, we use D'Alembert's solution in conjunction with Fourier series. D'Alembert's solution gives

$$u(x,t) = f(x-ct) + g(x+ct)$$

and we have to determine the functions f, g *so that the initial and boundary conditions are satisfied. Since* $u(0,t) = 0, \ u(l,t) = 0 \ \forall t \geq 0$ *we require*

$$f(-ct) + g(ct) = 0, \quad f(l-ct) + g(l+ct) = 0, \quad \forall t \geq 0$$

We rewrite these equations as

$$f(-z) + g(z) = 0, \quad f(l-z) + g(l+z) = 0$$

and assume that they hold for all $z \in \mathbf{R}$. *Replacing* z *by* $z + l$ *in the second of these equations we see that* $f(-z)+g(z+2l) = 0$ *and taken in conjunction with the first equation we conclude that* $g(z) = g(z + 2l)\ \forall z \in \mathbf{R}$. *In other words* g *is periodic with period* $2l$. *We also note that* $g(z) = g(z + 2l) = -f(-z)$ *so that* $g(z) = -f(-z)$. *Similarly, we can show that* f *is periodic with period* $2l$, $f(z) = f(z + 2l)\ \forall z \in \mathbf{R}$.

The fact that f *is a periodic function of period* $2l$ *suggests that we expand it as a Fourier series:*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/l) + \sum_{n=0}^{\infty} b_n \sin(n\pi x/l).$$

The corresponding Fourier series for g *is easily obtained from the relation* $g(z) = -f(-z)$*:*

$$g(x) = -\frac{1}{2}a_0 - \sum_{n=1}^{\infty} a_n \cos(n\pi x/l) + \sum_{n=0}^{\infty} b_n \sin(n\pi x/l).$$

We must also satisfy the initial condition $\frac{\partial u}{\partial t}(x, 0) = 0$ *from which we immediately derive the requirement* $-cf'(x)+cg'(x) = 0,\quad f'(x) = g'(x)$. *Formal differentiation of our Fourier series for* f, g *gives*

$$f'(x) = \sum_{n=1}^{\infty} a_n(-n\pi/l) \sin(n\pi x/l) + \sum_{n=1}^{\infty} b_n(n\pi/l) \cos(n\pi x/l),$$

$$g'(x) = \sum_{n=1}^{\infty} a_n(n\pi/l) \sin(n\pi x/l) + \sum_{n=1}^{\infty} b_n(n\pi/l) \cos(n\pi x/l)$$

Since $f'(x) = g'(x)$ *we see that*

$$\sum_{n=1}^{\infty} a_n(n\pi/l) \sin(n\pi x/l) = 0$$

from which we conclude that the a_n *are all zero. Our expressions for* f, g *can now be written*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin(n\pi x/l), \quad g(x) = -\frac{1}{2}a_0 + \sum_{n=1}^{\infty} b_n \sin(n\pi x/l).$$

Hence

$$u(x, t) = f(x - ct) + g(x + ct) = \sum_{n=1}^{\infty} b_n(\sin(n\pi(x - ct)/l) + \sin(n\pi(x + ct)/l).$$

Writing $B_n = 2b_n$ *we have*

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/l) \cos(n\pi ct/l).$$

Finally, we impose the condition $u(x,0) = \phi(x),\ x \in [0,l]$. *This requires*

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/l).$$

This is a Fourier sine series for $\phi(x)$ *and the* B_n *are just the Fourier coefficients with respect to the orthogonal functions* $\{\sin(n\pi x/l)\}, n = 1, 2, 3, \ldots$. *There is no need to remember the formulae for Fourier coefficients; formally we just multiply both sides by* $\sin(m\pi x/l)$ *and integrate with respect to* x *from* 0 *to* l, *using the orthogonality relation*

$$\int_0^l \sin(n\pi x/l)\sin(m\pi x/l)\,dx = \delta_{n,m}(l/2).$$

This immediately gives

$$\int_0^l \phi(x)\sin(m\pi x/l)\,dx = B_m(l/2)$$

whence

$$B_m = \frac{2}{l}\int_0^l \phi(x)\sin(m\pi x/l)\,dx.$$

The final solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/l)\cos(n\pi ct/l),$$

where the B_n *have just been determined.*

The solution which we have found has a physical interpretation. The total vibration is a superposition of terms $u_n(x,t) = B_n \sin(n\pi x/l)\cos(n\pi ct/l)$ each of which defines a vibration whose angular frequency ω_n is given by $\omega_n = n\pi c/l$; the corresponding frequency is $\nu_n = \omega_n/(2\pi) = nc/(2l)$. $n = 1$ defines the *fundamental* mode of vibration and each $n > 1$ defines a *harmonic*. We see that the n-th harmonic has frequency n times that of the fundamental. From our earlier work we know that $c^2 = T/\rho$, where T denotes the tension in the string and ρ its linear density, assumed constant. It follows that the permitted frequencies are each proportional to the square root of the tension. The fundamental, corresponding to $n = 1$, has frequency $c/2l = \frac{1}{2l}\sqrt{\frac{T}{\rho}}$.

We now describe the method of separation of variables which is frequently of great value in constructing solutions of partial differential equations.

12.5 Separation of variables

Let's consider this method in relation to the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

subject to the conditions:

$$u(x,0) = \phi(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in [0,l], \quad u(0,t) = 0, u(l,t) = 0 \ \ \forall t \geq 0 \tag{12.22}$$

The method which we describe will work for other equations such as the Laplace equation, the diffusion equation etc. which we consider later on.

We seek solutions of the form $u = \widetilde{u}(x,t)$ where $\widetilde{u}(x,t) = X(x)T(t)$. In other words, we are looking for solutions which can be written as the product of a function of x only multiplied by a function of t only. Substitution of the trial solution into the PDE gives

$$\frac{\partial^2(XT)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2(XT)}{\partial t^2}, \qquad T\frac{d^2X}{dx^2} = \frac{X}{c^2}\frac{d^2T}{dt^2}, \qquad \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2}.$$

Observe that the left handside of the third of these equations depends on x only whilst the right handside of this equation depends on t only. Since x, t are independent variables consistency demands that

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = \text{constant} \tag{12.23}$$

Let's look at the various possibilities. First take constant = 0 in equations 12.23. We immediately recover

$$X = ax + b, \qquad T = et + f \tag{12.24}$$

$$\widetilde{u} = (ax+b)(et+f) \tag{12.25}$$

where a, b, e, f are constants.

Next suppose that constant $= -\omega^2, \ \ \omega^2 > 0$. Solving 12.23 for X, T gives

$$X = A\cos\omega x + B\sin\omega x, \qquad T = D\cos\omega ct + E\sin\omega ct \tag{12.26}$$

$$\widetilde{u} = (A\cos\omega x + B\sin\omega x)(D\cos\omega ct + E\sin\omega ct) \tag{12.27}$$

where A, B, D, E are constants.

The third possibility is that constant $= \Omega^2, \ \ \Omega^2 > 0$. Solving 12.23 for X, T gives

$$X = F\cosh\Omega x + G\sinh\Omega x, \qquad T = H\cosh\Omega ct + K\sinh\Omega ct \tag{12.28}$$

$$\widetilde{u} = (F\cosh\Omega x + G\sinh\Omega x)(H\cosh\Omega ct + K\sinh\Omega ct) \tag{12.29}$$

where F, G, H, K are constants.

We see that the technique of *separation of variables* has generated a massive number of solutions of our PDE (we get a solution for every choice of the above parameters).

Moreover, since the wave equation is linear and homogeneous any linear combination of solutions will also be a solution. We guess that the harmonic solutions (the ones involving trigonometric functions are the ones most likely to be useful in solving the problem in hand (see 12.27). We have to remember that so far none of our solutions actually satisfies the initial and boundary conditions of the problem. To see how we can achieve this let's start with the solution 12.27

$$\widetilde{u} = (A\cos\omega x + B\sin\omega x)(D\cos\omega ct + E\sin\omega ct).$$

We want our solution to satisfy the condition $\frac{\partial u}{\partial t}(x,0) = 0\ \forall x \in [0,l]$. $\widetilde{u}$ will satisfy this if we choose $E = 0$. (We ignore the trivial prescription of setting A, B to zero). This gives

$$\widetilde{u} = (A\cos\omega x + B\sin\omega x)(D\cos\omega ct) \equiv (A'\cos\omega x + B'\sin\omega x)\cos\omega ct,$$

where $A' = AD,\ B' = BD$. We would also like our solution to satisfy $u(0,t) = 0\ \forall t \geq 0$ and this will be achieved if we choose $A' = 0$. This leaves us with a $\widetilde{u}$ given by

$$\widetilde{u} = (B'\sin\omega x)\cos\omega ct,$$

Our solution must also satisfy the condition $u(l,t) = 0\ \forall t \geq 0$. Apart from the trivial choice $B' = 0$ (which we ignore) we are compelled to choose ω such that $\omega l = n\pi$, where n is an integer. We can restrict ourselves to $n = 1, 2, 3, \ldots$ since $n = 0$ gives a trivial solution and the solution corresponding to $n = -N$ ($N > 0$) is just the negative of the solution corresponding to $n = N$. (sin is an odd function) Since we have a solution for each $n = 1, 2, 3, \ldots$ let's denote our solutions by $\widetilde{u_n}$ where

$$\widetilde{u_n}(x,t) = B_n \sin(n\pi x/l)\cos(n\pi ct/l).$$

The solutions $\widetilde{u_n}(x,t)$ which we have found satisfy three of the four conditions of the problem but it is clear that none of them can individually satisfy the fourth condition, namely $u(x,0) = \phi(x),\ x \in [0,l]$. Still, all is not lost! Our equation is linear homogeneous so any linear combination of the $\widetilde{u_n}(x,t)$ will provide a solution. This suggests that we try, as the solution to our problem,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/l)\cos(n\pi ct/l).$$

In order to satisfy the condition $u(x,0) = \phi(x),\ x \in [0,l]$. we must choose the B_n so that

$$\phi(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/l).$$

We recognize that this is just a Fourier sine series for $\phi(x)$ and the Fourier method as described above gives

$$B_n = \frac{2}{l}\int_0^l \phi(x)\sin(n\pi x/l)\,dx.$$

The solution to the problem is therefore

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/l)\cos(n\pi ct/l), \qquad B_n = \frac{2}{l}\int_0^l \phi(x)\sin(n\pi x/l)\,dx.$$

The reader will recognize that the solution obtained is identical to what we obtained previously by a different argument based on D'Alembert's solution. In fact the two methods which we have considered led to the development of Fourier series. The suggestion that a fairly arbitrary function could be expanded as a series of sines and cosines caused tremendous uproar among mathematicians when it was first proposed and many thought that it was impossible!

We don't pretend that we have dealt with every mathematical aspect of this problem – far from it. (For example, how do we know that the solution we've constructed satisfies the wave equation? It's not clear that formal differentiation of our solution will be valid; these are hard problems which we cannot investigate in this course.) As an application of our result consider the following particular case.

Example 12.4 *Consider a string stretched to a tension T whose ends are fixed at $x = 0$ and $x = 2l$. Suppose that the string is pulled aside through a distance h (small compared with l) and then released from rest. Find the displacement of the string at any subsequent time t.*

The answer is given above:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/2l)\cos(n\pi ct/2l), \qquad B_n = \frac{1}{l}\int_0^{2l} \phi(x)\sin(n\pi x/2l)\,dx.$$

(We've replaced l by $2l$, of course)

Here ϕ just gives the initial shape of the string and reference to a diagram will soon convince you that

$$\phi(x) = \begin{cases} hx/l & 0 \le x \le l \\ h(2l - x)/l & l \le x \le 2l \end{cases}$$

We have

$$\frac{l^2}{h} B_n = \int_0^l x\sin(n\pi x/2l)\,dx + \int_l^{2l} (2l - x)\sin(n\pi x/2l)\,dx$$

We can evaluate these two integrals by parts. We find

$$\int_0^l x\sin(n\pi x/2l)\,dx = -\frac{2l^2}{n\pi}\cos(n\pi/2) + \frac{4l^2}{n^2\pi^2}\sin(n\pi/2),$$

$$\int_l^{2l} (2l - x)\sin(n\pi x/2l)\,dx = \frac{2l^2}{n\pi}\cos(n\pi/2) + \frac{4l^2}{n^2\pi^2}\sin(n\pi/2),$$

which gives the result

$$B_n = \frac{8h}{n^2\pi^2}\sin(n\pi/2).$$

We see that B_n *is zero when* n *is even; only odd values of* n, $n = (2r+1)$, $r = 0, 1, 2, 3, \ldots$ *make a non–zero contribution. Substitution gives the displacement* $u(x,t)$ *as*

$$u(x,t) = \sum_{r=0}^{\infty} \frac{8h}{(2r+1)^2\pi^2}\sin((2r+1)\pi/2)\sin((2r+1)\pi x/2l)\cos((2r+1)\pi ct/2l)$$

We note that the fundamental freqency, corresponding to $r = 0$ *is* $\nu_0 = c/4l$. *Notice also that the amplitude of the higher harmonics is dampened by the factor* $1/(2r+1)^2$.

We recall that $u(l,0) = h$ *(initially the string was pulled back through a distance* h *at its mid–point). Plugging this into our solution we obtain the celebrated formula*

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8},$$

from which another famous formula is easily derived:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

12.6 Plane wave solutions

In this section we quickly review some basic jargon concerning plane harmonic waves with which all students of applied mathematics should be familiar.

First, note that the wave equation in one space dimension has solutions of the form

$$u = A\cos(k(x-ct)+\epsilon),$$

where A is regarded as positive. (recall D'Alembert's solution)

Sometimes it is convenient to consider the sine wave solution $A\sin(k(x-ct)+\epsilon)$ or perhaps the complex form

$$A\cos(k(x-ct)+\epsilon) + iA\sin(k(x-ct)+\epsilon) = A\exp(ik(x-ct)+i\epsilon) \equiv \widetilde{A}\exp(ik(x-ct))$$

where $\widetilde{A} = A\mathrm{e}^{i\epsilon}$.

In each case the wave moves with speed c parallel to the x–axis.

Whichever of these three forms we choose to work with we note that the wave repeats itself at regular space intervals λ where, for example, $\cos(k(x+\lambda-ct)+\epsilon) = \cos(k(x-ct)+\epsilon)$ so that

$$k\lambda = 2\pi, \qquad \lambda = \frac{2\pi}{k}.$$

λ is called the *wavelength* of the wave.

The time τ taken for one complete wave to pass any point is the *period* of the wave and is given by $\cos(k(x - c(t + \tau)) + \epsilon) = \cos(k(x - ct) + \epsilon)$ so that

$$kc\tau = 2\pi, \qquad \tau = \frac{2\pi}{kc} = \frac{\lambda}{c}, \qquad \text{so that } \lambda = c\tau.$$

The *frequency* ν of the wave is the number of waves passing a fixed point per unit time so that

$$\nu = \frac{1}{\tau} = \frac{kc}{2\pi}, \qquad \text{and } \lambda = \frac{c}{\nu}.$$

The quantity ϵ is called the *phase* or the *epoch* of the wave. Only phase differences have physical significance; two waves which are out of phase may interfere destructively.

A is called the *amplitude* of the wave. If we work with the complex exponential form $\widetilde{A}\exp(ik(x - ct))$ we have to bear in mind that the amplitude is given by $|\widetilde{A}|$ and the phase by $\arg \widetilde{A}$.

12.7 Waves with spherical symmetry

The three–dimensional wave equation has the form

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

Suppose that we are interested in solutions with spherical symmetry so that u depends only on the variables r, t, where $r = \sqrt{(x^2 + y^2 + z^2)}$. In this case

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru).$$

This follows either from the expression for Laplace's operator contained in the Appendix or, by direct calculation, as follows.

Since $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{\partial u}{\partial r}, \quad \frac{\partial^2 u}{\partial x^2} = \left(\frac{x}{r}\right)^2 \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\left(\frac{1}{r} - \frac{x}{r^2}\frac{x}{r}\right),$$

with two similar expressions for the second order derivatives of u with respect to y, z obtained by replacing x by y and z in turn. Adding the three expressions we obtain

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{3}{r}\frac{\partial u}{\partial r} - \frac{r^2}{r^3}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru),$$

where we have used the fact that $r^2 = x^2 + y^2 + z^2$.

Substituting this expression for $\nabla^2 u$ in the wave equation we obtain

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial^2}{\partial r^2}(ru) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(ru).$$

We recognize that this is just the one–dimensional wave equation for (ru) in terms of the variables r, t so we can write down D'Alembert's solution in the form

$$ru(r,t) = f(r-ct) + g(r+ct), \qquad u(r,t) = \frac{f(r-ct)}{r} + \frac{g(r+ct)}{r},$$

where f, g are arbitrary $\mathbf{C^2}$ functions. The first term in this expression for $u(r,t)$ represents a spherical wave moving away from the origin with speed c, whilst the second term represents a sperical wave converging on the origin (the centre of the spherical symmetry) with speed c.

We end this chapter with an example of the use of the technique of separation of variables in a problem involving the wave equation in three space dimensions.

Example 12.5 *Consider the wave equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\sigma^2}\frac{\partial^2 u}{\partial t^2}$$

inside the box $\{(x,y,z) : 0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c\}$. *Use the method of separation of variables and the Fourier method to find the solution which satisfies the following conditions:*

$$u(0,y,z,t) = 0,\ u(a,y,z,t) = 0,\ u(x,0,z,t) = 0,\ u(x,b,z,t) = 0,$$

$$u(x,y,0,t) = 0,\ u(x,y,c,t) = 0,$$

$$u(x,y,z,0) = f(x,y,z), \quad \frac{\partial u}{\partial t}(x,y,z,0) = 0.$$

Given the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\sigma^2}\frac{\partial^2 u}{\partial t^2}$$

seek solutions of the form $\tilde{u}$, where $\tilde{u}(x,y,z,t) = X(x)Y(y)Z(z)T(t)$. Separation of variables gives

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{\sigma^2 T}\frac{d^2T}{dt^2}$$

and

$$\frac{1}{\sigma^2 T}\frac{d^2T}{dt^2} = -\alpha^2, \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = -\beta^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -\gamma^2, \quad \frac{1}{X}\frac{d^2X}{dx^2} = -\delta^2, \quad \alpha^2 = \beta^2 + \gamma^2 + \delta^2.$$

We see that

$$\widetilde{u} = (A\cos\delta x + B\sin\delta x)(C\cos\gamma y + D\sin\gamma y)(E\cos\beta z + F\sin\beta z)(G\cos\alpha\sigma t + H\sin\alpha\sigma t)$$

satisfies the PDE. In order to satisfy the boundary conditions $\widetilde{u} = 0$ when $x = 0$, $x = a,\ y = 0, y = b,\ z = 0, z = c$ we must choose

$$A = 0,\ C = 0,\ E = 0,$$

$$\delta a = m_1\pi, \quad m_1 = 1, 2, 3, \ldots$$

$$\gamma b = m_2\pi, \quad m_2 = 1, 2, 3, \ldots$$

$$\beta c = m_3\pi, \quad m_3 = 1, 2, 3, \ldots$$

We therefore obtain a class of solutions of the form

$$\widetilde{u}_{m_1m_2m_3}(x, y, z, t) = \sin\frac{m_1\pi x}{a}\sin\frac{m_2\pi y}{b}\sin\frac{m_3\pi z}{c}\Phi(t),$$

where

$$\Phi(t) = (\widetilde{G}_{m_1m_2m_3}\cos(\alpha_{m_1m_2m_3}\sigma t) + \widetilde{H}_{m_1m_2m_3}\sin(\alpha_{m_1m_2m_3}\sigma t))$$

and

$$\alpha_{m_1m_2m_3} = \left(\frac{{m_1}^2\pi^2}{a^2} + \frac{{m_2}^2\pi^2}{b^2} + \frac{{m_3}^2\pi^2}{c^2}\right)^{1/2} \tag{12.30}$$

These solutions will satisfy the condition $\frac{\partial u}{\partial t}(x, y, z, 0) = 0$ if we take $\widetilde{H}_{m_1m_2m_3} = 0$. Applying the superposition principle we try to satisfy the remaining condition $u(x, y, z, 0) = f(x, y, z)$ with

$$u = \sum_{m_1,m_2,m_3=1}^{\infty}\widetilde{G}_{m_1m_2m_3}\sin\frac{m_1\pi x}{a}\sin\frac{m_2\pi y}{b}\sin\frac{m_3\pi z}{c}\cos(\alpha_{m_1m_2m_3}\sigma t) \tag{12.31}$$

This demands that

$$f(x, y, z) = \sum_{m_1,m_2,m_3=1}^{\infty}\widetilde{G}_{m_1m_2m_3}\sin\frac{m_1\pi x}{a}\sin\frac{m_2\pi y}{b}\sin\frac{m_3\pi z}{c}.$$

We use the Fourier method and multiply both sides by

$$\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\sin\frac{r\pi z}{c}$$

and carry out integrations $\int_0^a dx,\ \int_0^b dy,\ \int_0^c dz$ using the orthognality relations

$$\int_0^a \sin\frac{r\pi x}{a}\sin\frac{s\pi x}{a}\,dx = \frac{a}{2}\delta_{r,s},$$

for any non-negative integers r, s. etc. We conclude that

$$\widetilde{G}_{pqr} = \frac{8}{abc}\int f(x, y, z)\sin\frac{p\pi x}{a}\sin\frac{q\pi y}{b}\sin\frac{r\pi z}{c}\,dxdydz \tag{12.32}$$

where the (volume) integral is taken through the space $0 \le x \le a,\ 0 \le y \le b,$ $0 \le z \le c$. The final solution, satisfying all the boundary conditions, is defined by equations 12.30, 12.31, 12.32.

Chapter 13

Examples on Separation of Variables

The method of separation of variables which we introduced in the previous chapter is an important technique. We illustrate the method further by a set of examples.

Example 13.1 *Use the method of separation of variables to find solutions of the PDE*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = u.$$

Notice that this is a linear homogeneous equation and therefore the superposition principle can be applied. Use the Fourier method to show that the solution of the equation in the semi-infinite strip $0 \leq x \leq \pi,\ y \geq 1,\ (x,y) \neq (\pi,1)$ *which satisfies the conditions* $u(0,y) = 0,\ u(\pi,y) = 0,\ u(x,1) = x,$ *is*

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \exp[(1+n^2)(1-y)] \sin nx.$$

Given the PDE

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = u$$

we seek solutions of the form $\tilde{u}(x,y) = X(x)Y(y)$*. We have*

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{1}{Y}\frac{dY}{dy} = 1.$$

Separation of variables gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2, \quad X = A'\cos\lambda x + B'\sin\lambda x$$

and

$$\frac{1}{Y}\frac{dY}{dy} = -(\lambda^2+1), \quad Y = C'\exp-(\lambda^2+1)y.$$

We have obtained a class of solutions of the given PDE of the form

$$\widetilde{u} = C'(A'\cos\lambda x + B'\sin\lambda x)[\exp -(\lambda^2+1)y] \equiv (A\cos\lambda x + B\sin\lambda x)\exp -(\lambda^2+1)y.$$

In order to satisfy the boundary conditions $\widetilde{u}(0,y) = 0$, $\widetilde{u}(\pi,y) = 0$ *we need* $A = 0$, $\lambda = n$, $(n = 1, 2, 3, \ldots)$. *We therefore have a class of solutions*

$$\{\widetilde{u}_n(x,y) = B_n[\exp -(n^2+1)y]\sin nx, \; n = 1, 2, 3, \ldots\}$$

which satisfy the PDE and the boundary conditions $\widetilde{u}_n(0,y) = 0$, $\widetilde{u}_n(\pi,y) = 0$.

Since the PDE is linear and homogeneous the superposition principle holds and we therefore try to fit the remaining boundary condition using the trial solution

$$u(x,y) = \sum_{n=1}^{\infty} B_n[\exp -(n^2+1)y]\sin nx.$$

Since $u(x,1) = x$ *we require*

$$x = \sum_{n=1}^{\infty} B_n[\exp -(n^2+1)]\sin nx.$$

Application of the Fourier method using the formula

$$\int_0^{\pi} \sin nx \sin mx \, dx = \frac{\pi}{2}\delta_{n,m}$$

(m, n positive integers) gives

$$B_n = \frac{2}{\pi}[\exp(n^2+1)]\int_0^{\pi} x\sin nx \, dx = \frac{2(-1)^{n+1}}{n}\exp(n^2+1),$$

following an integration by parts. The final result is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}[\exp(n^2+1)(1-y)]\sin nx.$$

Example 13.2 *Show that the solution of the diffusion equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}$$

subject to the conditions

$$u(0,t) = 0, \quad u(1,t) = 1, \quad \forall t > 0; u(x,0) = 0, \quad 0 \le x \le 1$$

is

$$u(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}[\exp(-n^2\pi^2\kappa t)]\sin n\pi x.$$

What happens as $t \to \infty$? Offer a physical interpretation of this problem. (Look at the derivation of the diffusion equation in Chapter 8 and think in terms of the given boundary conditions.)

We have to solve

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}, \text{ subject to } u(0,t) = 0,\ u(1,t) = 1\ \forall t > 0;\ u(x,0) = 0,\ 0 \le x \le 1.$$

Using the method of separation of variables, writing $u = \tilde{u} = X(x)T(t)$, we obtain, in the usual way,

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0,\ \frac{dT}{dt} = -\kappa\lambda^2 T.$$

The choice $\lambda = 0$ gives $X(x) = a' + b'x$ whilst for $\lambda \neq 0$ we have $X(x) = A'\cos\lambda x + B'\sin\lambda x,\ T(t) = C'\mathrm{e}^{-\lambda^2\kappa t}$. The superposition principle (for linear homogeneous equations any linear conbination of solutions is a solution) now shows that

$$\tilde{u}(x,t) = ax + b + \sum_{\lambda\in\Lambda}(A_\lambda\cos\lambda x + B_\lambda\sin\lambda x)\mathrm{e}^{-\lambda^2\kappa t},\ \Lambda \text{ some index set},$$

is also a solution. In order to make $\tilde{u}(0,t) = 0\ \forall t > 0$ we take all the A_λ and b to be zero. In order to satisfy $\tilde{u}(1,t) = 1\ \forall t > 0$ we can choose $a = 1$ and $\lambda = n\pi,\ n = 1,2,3,\ldots$. On this basis we are led to consider as the solution to our problem

$$\tilde{u}(x,t) = x + \sum_{n=1}^{\infty} B_n \sin(n\pi x)\mathrm{e}^{-n^2\pi^2\kappa t}.$$

In order to satisfy the remaining condition $u(x,0) = 0\ \forall x \in [0,1]$ we require the B_n to satisfy

$$-x = \sum_{n=1}^{\infty} B_n\sin(n\pi x).$$

The Fourier method gives (the functions $\{\sin(n\pi x)\}$ are orthogonal over $[0,1]$)

$$B_m = -2\int_0^1 x\sin(m\pi x)\,dx = \frac{2(-1)^m}{m\pi},$$

following an integration by parts. We therefore obtain, as the solution of the PDE satisfying the boundary conditions,

$$u(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x)\mathrm{e}^{-n^2\pi^2\kappa t}.$$

As $t \to \infty$ RHS $\to x$*; this is intuitively clear since* $\mathrm{e}^{-n^2\pi^2\kappa t} \to 0$ *as* $t \to \infty$*. A more rigorous justification is achieved by the following argument: Write*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x)\mathrm{e}^{-n^2\pi^2\kappa t} = \sum_{n=1}^{m} \frac{(-1)^n}{n} \sin(n\pi x)\mathrm{e}^{-n^2\pi^2\kappa t} + R_m$$

where R_m*, the remainder after* m *terms, is given by*

$$R_m = \sum_{n=m+1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x)\mathrm{e}^{-n^2\pi^2\kappa t}.$$

It's clear that $\sum_{n=1}^{m} \to 0$ *as* $t \to \infty$ *since* $\mathrm{e}^{-n^2\pi^2\kappa t} \to 0$ *as* $t \to \infty$ *and in this case we're dealing with only a finite sum. As regards* R_m *note that*

$$|R_m| \leq \sum_{n=m+1}^{\infty} \frac{1}{n}\mathrm{e}^{-n^2\pi^2\kappa t} \leq \frac{1}{m+1} \sum_{n=m+1}^{\infty} \mathrm{e}^{-n^2\pi^2\kappa t} \quad (\text{using } |\sin \mathrm{n}\pi\mathrm{x}| \leq 1).$$

Now, $\mathrm{e}^{n^2\pi^2\kappa t} > \mathrm{e}^{n\pi^2\kappa t}$ *and therefore* $\mathrm{e}^{-n^2\pi^2\kappa t} < \mathrm{e}^{-n\pi^2\kappa t}$ *We conclude that*

$$|R_m| \leq \frac{1}{m+1} \sum_{n=m+1}^{\infty} \mathrm{e}^{-n\pi^2\kappa t} = \frac{1}{m+1} \frac{\mathrm{e}^{-(m+1)\pi^2\kappa t}}{1 - \mathrm{e}^{-\pi^2\kappa t}} \to 0 \text{ as } t \to \infty.$$

We conclude that $u(x,t) \to x$ *as* $t \to \infty$*.*

The physical interpretation is as follows. Imagine a heat conducting bar which lies on the $x-$*axis between* $x = 0$ *and* $x = 1$*. At* $t = 0$ *the bar is at zero temperature. The end* $x = 1$ *is then clamped and held at temperature 1 and the end* $x = 0$ *is held at zero temperature. The heat diffuses along the bar, raising the temperature according to the above formula, and after a long time tends to the uniform temperature distribution* $u = x$*; for large time the temperature increases linearly from zero at* $x = 0$ *to 1 at* $x = 1$*. This agrees with our intuition.*

Example 13.3 *In plane polar coordinates* (r, θ) *Laplace's operator is given by (see Appendix)*

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}.$$

Find solutions of Laplace's equation by writing $\phi(r,\theta) = R(r)\Theta(\theta)$ *and employing the method of separation of variables. Use your results to solve the following problem.*

Show that the solution $\phi(r,\theta)$ *of Laplace's equation in the semi-circular region* $r < a$*,*

$0 < \theta < \pi$, which vanishes on the line $\theta = 0$ and takes the constant value A on the line $\theta = \pi$ and on the curved boundary $r = a$, is

$$\phi(r,\theta) = \frac{A}{\pi}\Big[\theta + 2\sum_{n=1}^{\infty}\Big(\frac{r}{a}\Big)^n \frac{\sin n\theta}{n}\Big].$$

(Strictly speaking, the origin and the point $r = a,\ \theta = 0$ should be excluded)

In plane polar coordinates (r,θ) Laplace's equation is

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

Writing $\phi(r,\theta) = R(r)\Theta(\theta)$ and separating the variables we find that

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0,$$

$$\frac{d^2\Theta}{d\theta^2} + \lambda^2\Theta = 0, \quad r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda^2 R = 0.$$

The Θ equation has solution

$$\Theta = A'\cos\lambda\theta + B'\sin\lambda\theta, \quad (\lambda^2 \neq 0), \quad \Theta = c'\theta + d' \quad (\lambda^2 = 0),$$

whilst the R equation is satisfied by $R = Kr^l$ provided $l(l-1) + l - \lambda^2 = 0$ i.e. $l = \pm\lambda$ if $\lambda^2 \neq 0$ and by $R = e' + f'\ln r$ if $\lambda^2 = 0$.

This means that

$$\widetilde{\phi}(r,\theta) = (e' + f'\ln r)(c'\theta + d') + \sum_{\lambda}(C'_\lambda r^\lambda + D'_\lambda r^{-\lambda})(A'_\lambda\cos\lambda\theta + B'_\lambda\sin\lambda\theta)$$

is a solution of Laplace's equation. In the given problem we choose $f' = 0,\ D'_\lambda = 0$ since $\ln r,\ r^{-\lambda}$ are singular at $r = 0$. We therefore consider a solution of the form

$$\widetilde{\phi}(r,\theta) = (c\theta + d) + \sum_{\lambda} r^\lambda(A_\lambda\cos\lambda\theta + B_\lambda\sin\lambda\theta).$$

We want $\widetilde{\phi}(r,\theta = 0) = 0,\ 0 < r < a$. This demands that $d = 0,\ A_\lambda = 0$. We therefore look at the solution

$$\widetilde{\phi}(r,\theta) = c\theta + \sum_{\lambda} B_\lambda r^\lambda \sin\lambda\theta).$$

The condition $\widetilde{\phi}(r,\theta = \pi) = A$ is satisfied if we choose λ to be an integer, $\lambda = n$, $n = 1, 2, 3, \ldots$ and $a\pi = A$ so that $c = A/\pi$. We therefore try to satisfy the final condition with a solution of the form

$$\widetilde{\phi}(r,\theta) = \frac{A\theta}{\pi} + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta).$$

We want $\widetilde{\phi}(r=a,\theta) = A,\ 0<\theta<\pi$ *i.e.*

$$A(1-\theta/\pi) = \sum_{n=1}^{\infty} B_n a^n \sin(n\theta).$$

The Fourier method gives

$$B_n a^n \frac{\pi}{2} = A\int_0^{\pi} (1-\theta/\pi)\sin(n\theta)\,d\theta = \frac{A}{n},$$

after an integration by parts. This gives

$$B_n = \frac{2}{\pi}\frac{A}{a^n n},$$

and a solution $\phi(r,\theta)$ *given by*

$$\phi(r,\theta) = \frac{A\theta}{\pi} + \frac{2A}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{\sin(n\theta)}{n}.$$

Example 13.4 *Laplace's equation in three–dimensional cylindrical coordinates* (r,θ,z) *takes the form (see Appendix)*

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

Find solutions of this equation by separation of variables.

Substitute $\phi = R(r)\Theta(\theta)Z(z)$ *and divide through by* $R\Theta Z$ *to obtain*

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

Consistency requires (remember: (r,θ,z) *are independent variables)*

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = p^2, \quad p^2>0, \quad \frac{d^2Z}{dz^2} - p^2Z = 0,$$

so that

$$Z(z) = A\cosh pz + B\sinh pz.$$

(We could equally well have written $-q^2,\ q^2>0$ *instead of* p^2*. Moreover, we could have set* $p^2=0$*, in which case* $Z(z)=a+bz$*) Our equation now has the form*

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + p^2r^2 = 0.$$

Since r, θ *are independent consistency demands that*

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -n^2, \quad n^2 > 0$$

with solution

$$\Theta(\theta) = C\cos n\theta + D\sin\theta.$$

Again, we could equally well have chosen to write $m^2,\ m^2 > 0$ *rather than* $-n^2$*. Furthermore,* $n^2 = 0$*, is allowable, in which case* $\Theta(\theta) = c\theta + d$*; there are problems in which this solution is useful. However, in most applications, where we are looking for solutions such that* $\Theta(0) = \Theta(2\pi)$*, our first choice is appropriate. Imposing the condition* $\Theta(0) = \Theta(2\pi)$ *we choose* n *to be an integer and we are left with the following equation for* R*:*

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (p^2r^2 - n^2)R = 0.$$

If we change the variable from r *to* $s = nr$ *the equation for* R *becomes*

$$s^2\frac{d^2R}{ds^2} + s\frac{dR}{ds} + (s^2 - n^2)R = 0 \tag{13.1}$$

This differential equation arises in many branches of mathematical physics and is called Bessel's equation. It leads to the study of the Bessel functions $J_n(s)$ *which satisfy Bessel's equation.* *(See Example 6.3, page 47)*

Chapter 14

Laplace's Equation

In this chapter we give a very brief introduction to Laplace's equation in three dimensions.

A basic problem in the theory of differential equations (both ordinary and partial) is (a) to prove the existence of a solution and (b) to prove (where possible) that if a solution exists it is unique. Generally speaking this is a difficult problem and so far we have paid no attention to matters of existence and uniqueness beyond obtaining explicit solutions to specific problems. In certain cases physical intuition indicates that the solution obtained is unique, but we have not given a mathematical proof. In the case of Laplace's equation we can prove the following uniqueness theorem.

Theorem 14.1 *Suppose that ϕ is a $\mathbf{C^2}$ function in a region of three-space which contains a volume V (assumed finite) bounded by a smooth closed surface S. Suppose that ϕ satisfies Laplace's equation throughout V and takes a prescribed set of boundary values on S (defined by some continuous function on S). Then ϕ is unique.*

Proof 1 *We write* grad *for* ∇ *i.e.* $\mathrm{grad} \equiv \nabla$.

Note that for any function u which satisfies Laplace's equation

$$\mathrm{div}(u\,\mathrm{grad}\,u) = u\nabla^2 u + \mathrm{grad}\,u.\,\mathrm{grad}\,u = |\,\mathrm{grad}\,u|^2.$$

By Gauss's theorem

$$\int_S u\,\mathrm{grad}\,u.\mathbf{n}\,dS = \int_V |\,\mathrm{grad}\,u|^2\,dV,$$

where $\mathbf{n}$ denotes the outward drawn unit normal to S.

Suppose now that there are two $\mathbf{C^2}$ functions ϕ_1 and ϕ_2 which satisfy the conditions of the theorem i.e. they satisfy Laplace's equation in V and take the same prescribed boundary values on S (defined by some continuous function on S). Put $u = \phi_1 - \phi_2$. Then u also satisfies Laplace's equation in V and takes the value zero on S. It follows from the above result that

$$\int_V |\,\mathrm{grad}\,u|^2\,dV = 0$$

Since the integrand is continuous and non–negative we deduce that $|\operatorname{grad} u|$ *is identically zero throughout the region of integration, and therefore* u *is constant throughout* V. *It follows that*

$$\phi_1 - \phi_2 = \text{constant in } V.$$

But ϕ_1 *and* ϕ_2 *coincide on* S *and therefore the constant is zero. In other words* $\phi_1 \equiv \phi_2$.

The boundary conditions referred to in the theorem are called *Cauchy* boundary conditions. In the case of *Neumann* boundary conditions we prescribe the normal derivative $\frac{\partial \phi}{\partial n} \equiv \operatorname{grad} \phi.\mathbf{n}$ as a continuous function on S. In the case of Neumann conditions (as opposed to Cauchy conditions) we can repeat the above argument, word for word, up to the point where we draw the conclusion that $\phi_1 - \phi_2 =$ constant in V. However, in the Neumann case we cannot deduce that the constant is zero; we can only show that the solution is unique up to a constant.

The above arguments show that one can prescribe Cauchy or Neumann boundary conditions — not both together.

Our uniqueness theorem readily extends to infinite regions and the same conclusions can be drawn — provided we make suitable assumptions about the behaviour of ϕ at infinity.

Example 14.1 *We shall see in a moment that the function* $\phi_1 = 1/r,\ r = (x^2+y^2+z^2)^{1/2}$ *satisfies Laplace's equation; clearly the same is true of* $\phi_2 \equiv 1$. *Obviously both* ϕ_1 *and* ϕ_2 *take the same value,* 1, *on the unit sphere, centre the origin and radius* 1. *On the face of it this flies in the face of our theorem. There is no contradiction, however;* ϕ_1 *is not* $\mathbf{C^2}$ *at the origin.*

14.1 Spherical solutions

We can easily find all the spherically symmetric solutions of Laplace's equation. These are the solutions of the form $u = u(r)$, where $r = (x^2+y^2+z^2)^{1/2}$. Reference to the Appendix for the relevant expression for Laplace's operator in spherical polar coordinates (r, θ, ψ), or to Chapter 6, shows that the spherically symmetric solutions of Laplace's equation are given by

$$\nabla^2 u = 0, \qquad \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = 0.$$

Integration with respect to r gives $(ru)' = C_1$ and a further integration yields

$$ru = C_1 r + C_2, \qquad u(r) = C_1 + \frac{C_2}{r} \tag{14.1}$$

(of course, C_1, C_2 are arbitrary constants). The solution 14.1 has a simple physical interpretation; it represents, for example, the elctrostatic potential due to a charge C_2 at the origin (or a constant times C_2, depending on the choice of units).

14.2 Further simple solutions

We note that

$$\frac{\partial^{m+n+p}}{\partial x^m \, \partial y^n \, \partial z^p}\Big(\frac{1}{r}\Big), \quad r \neq 0 \tag{14.2}$$

is also a solution of Laplace's equation for any choice of the non–negative integers m, n, p. For,

$$\nabla^2\Big(\frac{\partial^{m+n+p}}{\partial x^m \, \partial y^n \, \partial z^p}\Big(\frac{1}{r}\Big)\Big) = \frac{\partial^{m+n+p}}{\partial x^m \, \partial y^n \, \partial z^p}\Big(\nabla^2 \frac{1}{r}\Big) = 0$$

by virtue of the commutative law of partial differentiation.

It's interesting to compute some particular cases of 14.2. For example $\frac{\partial}{\partial z}(1/r)$, $\frac{\partial^2}{\partial z^2}(1/r)$ are solutions of Laplace's equation. Since

$$r^2 = x^2 + y^2 + z^2, \;\; 2r\frac{\partial r}{\partial z} = 2z, \;\; \frac{\partial r}{\partial z} = \frac{z}{r}$$

we find that

$$\frac{\partial}{\partial z}(1/r) = -\frac{z}{r^3}, \qquad \frac{\partial^2}{\partial z^2}(1/r) = \frac{3z^2}{r^5} - \frac{1}{r^3}.$$

Expressing z in terms of the usual spherical polar coordinates (r, θ, ψ), $z = r\cos\theta$, we see that

$$\frac{-\cos\theta}{r^2}, \quad \frac{3\cos^2\theta - 1}{r^3}$$

are solutions of Laplace's equation. Students of electrostatics will recognize that they correspond to the elctrostatic potential due to a certain elctrostatic dipole and quadrupole, respectively, at the origin $r = 0$.

By looking at the n–th derivative of $1/r$ we come to the conclusion that Laplace's equation has solutions of the form $r^{-(n+1)}P_n(\cos\theta)$, where (r, θ, ψ) are spherical polar coordinates and the P_n are polynomials of degree n in $\cos\theta$ which can be identified as the Legendre polynomials. The Legendre polynomials are orthogonal in the sense that $\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$ $(m \neq n)$ and have many other interesting properties; however, we do not consider them further in this course.

Chapter 15

Fourier Transforms and some applications

In Chapter 12 we indicated that for a suitably smooth function f defined on on $[-l, l]$ the Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{l}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{l}),$$

where the Fourier coefficients $a_n,\ b_n$ are defined by

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos(n\pi x/l)\,dx, \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin(n\pi x/l)\,dx \qquad n = 0, 1, 2, 3\ldots$$

converges to $\frac{1}{2}(f(x+0) + f(x-0))$ i.e. to $f(x)$ if x is a point of continuity of f.

Suppose now that f is a function which is absolutely integrable i.e. $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$ and that it satisfies the conditions stated in Chapter 12 for arbitrarily large l. The following is a heuristic argument which suggests a result which may be true, subject to suitable assumptions about the function f. Suppose that x is a point of continuity of f and write $l = \pi\lambda$. We then have, inserting the relevant expressions for the Fourier coefficients a_n, b_n in terms of f,

$$f(x) = \frac{1}{2\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda} f(y)\,dy + \frac{1}{\pi\lambda}\sum_{n=1}^{\infty}\int_{-\pi\lambda}^{\pi\lambda} f(y)\Big[\cos\Big(\frac{ny}{\lambda}\Big)\cos\Big(\frac{nx}{\lambda}\Big) + \sin\Big(\frac{ny}{\lambda}\Big)\sin\Big(\frac{nx}{\lambda}\Big)\Big]\,dy$$

$$= \frac{1}{2\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda} f(y)\,dy + \frac{1}{\pi\lambda}\sum_{n=1}^{\infty}\int_{-\pi\lambda}^{\pi\lambda} f(y)\cos\frac{n}{\lambda}(y-x)\,dy.$$

Now suppose λ is large and positive. Write $\alpha = \frac{n}{\lambda},\quad \delta\alpha = \frac{1}{\lambda}$ and think of a definition of the integral in terms of Riemann sums. Since we are assuming that f is absolutely integrable it follows that $\frac{1}{2\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda} f(y)\,dy \to 0$ as $\lambda \to \infty$ and it becomes plausible that

$$f(x) = \frac{1}{\pi}\int_0^{\infty} d\alpha \int_{-\infty}^{\infty} f(y)\cos\alpha(y-x)\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} f(y)\cos\alpha(y-x)\,dy.$$

This formula is known as Fourier's integral formula.

Since $\int_{-\infty}^{\infty} f(y)\sin\alpha(y-x)\,dy$ is an *odd* function of α we take it for granted that

$$0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} f(y)\sin\alpha(y-x)\,dy.$$

Combining this expression with Fourier's integral formula we are led to write

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} f(y)e^{i\alpha(y-x)}\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x} d\alpha \int_{-\infty}^{\infty} e^{i\alpha y} f(y)\,dy. \tag{15.1}$$

For suitable f we define the Fourier transform of f, to be the function $\widetilde{f}$ defined by

$$\widetilde{f}(x) = \int_{-\infty}^{\infty} e^{ixy} f(y)\,dy \tag{15.2}$$

Note that the integral defining $\widetilde{f}$ certainly exists if f is absolutely integrable. If we imagine that $\widetilde{f}$ is given, and that f is unknown, we can recover f from equation (15.1) to obtain

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\widetilde{f}(\alpha)\,d\alpha \tag{15.3}$$

which is the so called inverse Fourier transform, and enables us to calculate f given $\widetilde{f}$. Since α is a dummy variable in this formula we can write any other symbol in lieu of α e.g. y.

Books on the theory of Fourier integrals, for example E.C. Titchmarsh, Theory of Fourier Integrals, investigate conditions on f which ensure the existence of the Fourier transform, and the validity of the corresponding inverse transform formula. This a hard branch of analysis and in this course we cannot proceed with such an investigation. However, it is not very difficult to outline an argument which justifies the above conclusions in the case of Schwartz[1]class functions on $\mathbf{R}$. The Schwartz class $\mathcal{S}(\mathbf{R})$ is the class of infinitely differentiable functions f on $\mathbf{R}$ which are of fast decrease at infinity; the second condition requires that for every choice of the integers m, N we have $|x^N f^{(m)}(x)| \to 0$ as $|x| \to \infty$.. In essence this means that a Schwartz class function f and all it derivatives tend to zero at infinity so fast as to dominate any power of x. A typical Schwartz class function is e^{-x^2}. For any Schwartz class function f we observe that the Fourier transform $\widetilde{f}$ of f always exists i.e. the integral $\int_{-\infty}^{\infty} e^{ixy} f(y)\,dy$ exists. Moreover, it is not hard to show rigorously that $\widetilde{f}$ is infinitely differentiable and is also Schwartz class.

As a prelude to verifying the Fourier inverse transform formula for Schwartz class functions we show how to compute integrals of the form $\int_{-\infty}^{\infty} e^{-\alpha(x+i\beta)^2}\,dx, \quad \alpha > 0, \quad \beta \in \mathbf{R}$ by integrating $e^{-\alpha z^2}$ round the rectangle whose vertices are at the points $z = -R$,

[1]Laurent Schwartz. Professor of Mathematics at Strasbourg. Gave a rigorous account of the Dirac δ-function in his book 'Theorie des Distributions' (Hermann, Paris (1957))

$z = R,\ z = R + i\beta,\ z = -R + i\beta$. Cauchy's theorem gives

$$\int_{-R}^{R} e^{-\alpha x^2}\,dx + \int_0^{\beta} e^{-\alpha(R+iy)^2}(i\,dy) + \int_R^{-R} e^{-\alpha(x+i\beta)^2}\,dx + \int_\beta^0 e^{-\alpha(-R+iy)^2}(i\,dy) = 0.$$

Now

$$\left|\int_0^\beta e^{-\alpha(R+iy)^2} i\,dy\right| \le \int_0^\beta \left|e^{-\alpha(R^2+2iRy-y^2)}\right| dy = e^{-\alpha R^2}\int_0^\beta e^{\alpha y^2}\,dy \to 0$$

as $R \to \infty$ Similarly we can show that $\int_\beta^0 e^{-\alpha(-R+iy)^2}(i\,dy)$ tends to zero in the limit $R \to \infty$. It follows that

$$\int_{-\infty}^{\infty} e^{-\alpha(x+i\beta)^2}\,dx = \int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}, \quad \alpha > 0, \quad \beta \in \mathbf{R}\,. \tag{15.4}$$

We may note in passing that by taking the real part of both sides of this formula we obtain

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\cos 2\alpha\beta x\,dx = e^{-\alpha\beta^2}\sqrt{\frac{\pi}{\alpha}}, \qquad \alpha > 0, \quad \beta \in \mathbf{R}\,.$$

15.1 Fourier Inverse Transform for Schwartz Class Functions

For $f \in \mathcal{S}(\mathbf{R})$ consider the inverse Fourier transform of $e^{-\epsilon x^2}\widetilde{f}(x), \quad \epsilon > 0$; ultimately we let $\epsilon \to 0$. We have

$$\int_{-\infty}^{\infty} e^{-\epsilon x^2}\widetilde{f}(x)e^{-ixz}\,dx = \int_{-\infty}^{\infty} e^{-\epsilon x^2}e^{-ixz}\,dx\Big(\int_{-\infty}^{\infty} f(y)e^{ixy}\,dy\Big) \tag{15.5}$$

$$= \int_{-\infty}^{\infty} f(y)\Big(\int_{-\infty}^{\infty} e^{-\epsilon[x^2+\frac{ix(z-y)}{\epsilon}]}\,dx\Big) \tag{15.6}$$

$$= \int_{-\infty}^{\infty} f(y)\,dy\int_{-\infty}^{\infty} e^{-\epsilon[\left(x+\frac{i(z-y)}{2\epsilon}\right)^2+\frac{(y-z)^2}{4\epsilon^2}]}\,dx = \sqrt{\frac{\pi}{\epsilon}}\int_{-\infty}^{\infty} f(y)e^{-\frac{(y-z)^2}{4\epsilon}} \tag{15.7}$$

$$= \sqrt{\frac{\pi}{\epsilon}}\Big[\int_{-\infty}^{\infty}[f(y)-f(z)]e^{-\frac{(y-z)^2}{4\epsilon}}\,dy + f(z)\int_{-\infty}^{\infty} e^{-\frac{(y-z)^2}{4\epsilon}}\,dy\Big] \tag{15.8}$$

(using formula (15.4))

$$= 2\pi f(z) + \sqrt{\frac{\pi}{\epsilon}}\int_{-\infty}^{\infty}[f(y)-f(z)]e^{-\frac{(y-z)^2}{4\epsilon}}\,dy \tag{15.9}$$

We now show that

$$\sqrt{\frac{\pi}{\epsilon}}\int_{-\infty}^{\infty}[f(y)-f(z)]e^{-\frac{(y-z)^2}{4\epsilon}}\,dy \to 0$$

as $\epsilon \to 0+$. To see this, we suppose that f is real valued (if not, we can apply the following argument to the real and imaginary parts of f separately) and note (by the Mean Value Theorem) that $f(y)-f(z) = f'(\xi(y))(y-z)$ for some number $\xi(y)$ (depending on y) between y and z. Since f is Schwartz class f' is certainly bounded and we write $K = \sup_{x\in\mathbf{R}} |f'(x)|$ to obtain

$$\left|\sqrt{\frac{\pi}{\epsilon}}\int_{-\infty}^{\infty}[f(y)-f(z)]e^{-\frac{(y-z)^2}{4\epsilon}}\,dy\right| \le \sqrt{\frac{\pi}{\epsilon}}K\int_{-\infty}^{\infty}|y-z|e^{-\frac{(y-z)^2}{4\epsilon}}\,dy$$

$$= \sqrt{\frac{\pi}{\epsilon}}(2K)\int_{0}^{\infty} ue^{-\frac{u^2}{4\epsilon}}\,du = 4K\sqrt{\pi\epsilon}$$

which does indeed tend to zero as ϵ tends to zero through positive values.

It's not very hard to prove rigorously that for Schwartz class f

$$\lim_{\epsilon\to 0+}\int_{-\infty}^{\infty} e^{-\epsilon x^2}\tilde{f}(x)e^{-ixz}\,dx = \int_{-\infty}^{\infty}\tilde{f}(x)e^{-ixz}\,dx,$$

We deduce that for any Schwartz class function f that

$$\int_{-\infty}^{\infty}\tilde{f}(x)e^{-ixz}\,dx = 2\pi f(z)$$

so that the inverse transform formula

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(x)e^{-ixz}\,dx, \qquad \forall f\in\mathcal{S}(\mathbf{R})$$

is established.

15.2 Examples

Example 15.1 *Find the Fourier transform of* $e^{-x^2/2}$.

Write $f(x) = e^{-x^2/2}$. Then

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-x^2/2}e^{ikx}\,dx = \int_{-\infty}^{\infty} e^{-(x^2-2ikx)/2}\,dx = e^{-k^2/2}\int_{-\infty}^{\infty} e^{-(x-ik)^2/2}\,dx = \sqrt{2\pi}e^{-k^2/2}.$$

using equation (15.4). We see that, apart from a multiplicative constant, the Fourier transform of f is f itself.

Example 15.2 *Suppose that*

$$f(x) = \begin{cases} e^{-x} & x\ge 0\\ 0 & x<0\end{cases}$$

Calculate the Fourier transform $\tilde{f}$ *of* f *and evaluate the inverse transform of* $\tilde{f}$ *when this is defined.*

Straightforward integration gives

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx = \int_{0}^{\infty} e^{x(ik-1)}\, dx = \frac{1}{1-ik}.$$

In order to compute the inverse transform of $\frac{1}{1-ik}$ we need to evaluate

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\widetilde{f}(k)\, dk = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{k+i}\, dk$$

We consider the two cases $x < 0,\ \ x > 0$ separately — note that the integral does not exist when $x = 0$. First note that with $k = \alpha + i\beta$ we have $-ikx = \beta x - i\alpha x$. This suggests that for $x > 0$ we can evaluate the integral by integrating $\frac{e^{-ikx}}{k+i}$ round the semi-circle centre $k = 0$ and radius R in the lower-half complex k-plane, whilst for $x < 0$ we should integrate this function round the corresponding semi-circle in the upper-half complex k-plane — these choices of contour will ensure that the exponential will be suitably damped for large R and guarantee that in the limit $R \to \infty$ the integral round the curved part of the contour tends to zero. (see the diagrams)

Standard arguments using Cauchy's residue theorem then show that

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{k+i}\, dk = \frac{(2\pi i)}{(2\pi i)} e^{-i(-i)x} = e^{-x}, \quad x > 0.$$

and

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{k+i}\, dk = 0, \quad x < 0$$

We see that the inverse transform of $\widetilde{f}$ is in fact equal to $f(x)$, except when $x = 0$. This is not entirely surprising, since $x = 0$ is a point of discontinuity of the function f.

In what follows we assume that the functions f which we consider are such that equations (15.2) and (15.3) defining the Fourier transform and inverse Fourier transform are valid.

Example 15.3 *Solve the diffusion equation*

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial \theta}{\partial t}, \quad x \in \mathbf{R}, \quad t \geq 0$$

subject to the conditions $\theta \to 0$, $\frac{\partial \theta}{\partial x} \to 0$ *as* $|x| \to \infty$, *and* $\theta(x,0) = f(x)$, $x \in \mathbf{R}$.

Multiply the given equation by e^{ixy} and integrate over x from $-\infty$ to ∞. This gives

$$\int_{-\infty}^{\infty} e^{ixy}\frac{\partial^2 \theta}{\partial x^2}\,dx = \frac{1}{\kappa}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \theta(x,t)e^{ixy}\,dx = \frac{1}{\kappa}\frac{\partial \widetilde{\theta}}{\partial t}$$

assuming that differentiation under the integral with respect to t is valid. Here $\widetilde{\theta}$ is the Fourier transform of θ, taken with respect to the variable x — so $\widetilde{\theta}$ depends on y and t i.e. $\widetilde{\theta} = \widetilde{\theta}(y,t)$. The trick now is to integrate the left hand side by parts and use the boundary conditions at infinity. This gives

$$\left[e^{ixy}\frac{\partial \theta}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (iy)e^{ixy}\frac{\partial \theta}{\partial x}\,dx \right] = \frac{1}{\kappa}\frac{\partial \widetilde{\theta}}{\partial t}$$

Using the condition $\frac{\partial \theta}{\partial x} \to 0$ as $|x| \to \infty$ we obtain, after a further partial integration,

$$(-iy)\left[e^{ixy}\theta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (iy)\theta e^{ixy}\,dx \right].$$

Since $\theta \to 0$ as $|x| \to \infty$ we obtain

$$-y^2\widetilde{\theta} = \frac{1}{\kappa}\frac{\partial \widetilde{\theta}}{\partial t} \qquad \widetilde{\theta} = A(y)e^{-\kappa y^2 t},$$

where A is independent of t but depends on y. Setting $t = 0$ and using the initial condition $\theta(x,0) = f(x)$ we derive

$$A = \widetilde{\theta}(y,0) = \int_{-\infty}^{\infty} \theta(x,0)e^{ixy} = \int_{-\infty}^{\infty} f(x)e^{ixy} = \widetilde{f}(y).$$

We therefore have $\widetilde{\theta} = e^{-\kappa y^2 t}\widetilde{f}(y)$. Applying the inverse Fourier transform we now derive

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy}\widetilde{\theta}(y,t)\,dy = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy}e^{-\kappa y^2 t}\widetilde{f}(y)\,dy.$$

Inserting the expression for the Fourier transform of f i.e.

$$\widetilde{f}(y) = \int_{-\infty}^{\infty} f(z)e^{izy}\,dz$$

we find that

$$\theta = \frac{1}{2\pi}\int e^{-ixy}e^{-\kappa y^2 t}\,dy\Big(\int_{-\infty}^{\infty} f(z)e^{izy}\,dz\Big)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\,dz\Big(\int_{-\infty}^{\infty} e^{iy(z-x)-\kappa y^2 t}\,dy\Big)$$

(upon changing the order of integration).

Now,

$$iy(z-x)-\kappa y^2 t = -\kappa t\Big[y^2+\frac{iy(x-z)}{\kappa t}\Big] = -\kappa t\Big[\Big(y+\frac{i(x-z)}{\kappa t}\Big)^2+\frac{(x-z)^2}{4\kappa^2 t^2}\Big].$$

We can now perform the y integral using equation (15.4) and finally obtain

$$\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)e^{-\frac{(x-z)^2}{4\kappa t}}\sqrt{\frac{\pi}{\kappa t}} = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} f(z)e^{-\frac{(x-z)^2}{4\kappa t}}\,dz.$$

If we assume that $f(x)=\delta(x)$, where δ is the Dirac[2] δ function, — physicists think of this as a spike function with the property that $\delta(x)$ is zero everywhere on the x-axis except at zero, where it is supposed to be so large that $\int_{-\infty}^{\infty}\delta(x)\,dx = 1$ — we obtain

$$\theta = \frac{1}{\sqrt{4\pi\kappa t}}e^{-\frac{x^2}{4\kappa t}} \tag{15.10}$$

a famous solution of the diffusion equation. Note that the θ given by equation (15.10) can be regarded as a probability density (since it has the property $\int_{-\infty}^{\infty}\theta(x,t)\,dx = 1$) which is initially concentrated at $x=0$ but subsequently diffuses along the x-axis.

As a second example on the use of Fourier transforms in solving partial differential equations consider the following:

Example 15.4 *Show that the solution of the two-dimensional Laplace equation*

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2} = 0, \quad x>0, \quad 0<y<b$$

subject to the conditions $V(0,y)=0,\quad 0<y\le b,\quad V(x,0)=f(x),\ x>0,$ $V(x,b)=0,\ x\ge 0,\quad V,\ \frac{\partial V}{\partial x}\to 0$ *as* $x\to\infty$ *is*

$$V(x,y) = \frac{2}{\pi}\int_0^{\infty} f(u)\,du\int_0^{\infty}\frac{\sinh k(b-y)}{\sinh(kb)}\sin(kx)\sin(ku)\,dk$$

At first sight this does not seem to be a problem where the use of Fourier transforms would be appropriate since $x\in[0,\infty)$ rather than $x\in(-\infty,\infty)$. However, we can extend the definition of V and f to $x\in(-\infty,\infty)$ by requiring V and f to be *odd functions* of x so that $V(-x,y)=-V(x,y)$ and $f(-x)=-f(x)$. This condition on V enables us to satisfy the requirement $V(0,y)=0,\quad 0<y\le b$ and since V now has meaning for

[2]Paul Dirac (1902-1984) Lucasian Professor of Mathematics at Cambridge. One of the founding fathers of Quantum Mechanics. Nobel Laureate (1933)

$x \in \mathbf{R}$ we are in a position to take Fourier transforms of V with respect to the variable x. Multiplying the Laplace equation by e^{ikx} and integrating with respect to x as in the preceding example gives

$$-k^2\widetilde{V} + \frac{\partial^2 \widetilde{V}}{\partial y^2} = 0,$$

where

$$\widetilde{V}(y,k) = \int_{-\infty}^{\infty} e^{ikx} V(x,y)\, dx.$$

Solving the equation for $\widetilde{V}$ gives

$$\widetilde{V} = Ae^{ky} + Be^{-ky}$$

Since $V(x,0) = f(x), \quad x \in \mathbf{R}$ we have $\widetilde{V}(0,k) = \widetilde{f}(k)$ and since $V(x,b) = 0$ we obtain $\widetilde{V}(b,k) = 0$. Forcing these conditions on $\widetilde{V}$ we derive the following equaions for A, B:

$$\widetilde{f} = A + B, \qquad 0 = Ae^{kb} + Be^{-kb}.$$

These linear algebraic equations are easily solved to give

$$A = -\frac{\widetilde{f}(k)e^{-kb}}{2\sinh(kb)}, \qquad B = \frac{\widetilde{f}(k)e^{kb}}{2\sinh(kb)}.$$

This leads to the following equation for $\widetilde{V}$:

$$\widetilde{V} = \frac{\widetilde{f}(k)\sinh k(b-y)}{\sinh(kb)}. \tag{15.11}$$

Taking the inverse transform we obtain

$$V(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\widetilde{V}(y,k)\, dk.$$

At this stage we recall that we extended the definition of V, f as *odd functions* of x. It is easy to check that this implies that both $\widetilde{V}$ and $\widetilde{f}$ are *odd functions* of k. We can therefore write

$$V(x,y) = \frac{1}{2\pi}\int_{0}^{\infty} e^{-ikx}\widetilde{V}(y,k) + \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ikx}\widetilde{V}(y,k)$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-ikx}\widetilde{V}(y,k) - \frac{1}{2\pi}\int_{0}^{\infty} e^{ikx}\widetilde{V}(y,k) = -\frac{i}{\pi}\int_{0}^{\infty}\sin(kx)\widetilde{V}(y,k)\, dk \tag{15.12}$$

Similarly, using the fact that we defined f on $\mathbf{R}$ as an *odd function* of x, we derive

$$\widetilde{f}(k) = 2i\int_{0}^{\infty}\sin(ku) f(u)\, du.$$

Substituting this expression for $\widetilde{f}$ into equation (15.11) and incorporating the resulting formula for $\widetilde{V}$ into equation (15.12) gives the stated result.

Before leaving the topic of Fourier transforms we note that the ideas developed above can be extended to functions f of n real variables $(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$. For suitable f we define the Fourier transform $\widetilde{f}$ of f by

$$\widetilde{f}(k_1, k_2, \ldots, k_n) = \int_{\mathbf{R}^n} e^{i\sum_{j=1}^n k_j x_j} f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n$$

the corresponding inversion formula being given by

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{-i\sum_{j=1}^n k_j x_j} \widetilde{f}(k_1, k_2, \ldots, k_n) dk_1 dk_2 \ldots dk_n.$$

Multivariable Fourier transforms arise in many areas of applied mathematics, for example in quantum mechanics.

Chapter 16

Problems

In the following questions the term *arbitrary function* occurs several times; the functions referred to are not totally arbitrary, of course, and must be differentiable to whatever order is necessary for the question to make sense.

Problem 1 *Eliminate the arbitrary function f in $u = f(x-y)$ to find a first order partial differential equation satisfied by u. Show, conversely, that this PDE has general solution $u = f(x-y)$.*

[Hint: Change to new variables α, β where $\alpha = x - y,\ \beta = x$.]

Problem 2 *Eliminate the arbitrary functions f, g in*

$$u(x,y) = f\left(\frac{x}{y}\right) + g(x-y)$$

by finding a partial differential equation satisfied by u.

Conversely, show that the partial differential equation

$$x\frac{\partial^2 u}{\partial x^2} + (x+y)\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

has general solution given by

$$u(x,y) = f\left(\frac{x}{y}\right) + g(x-y).$$

[Hint: Change to new variables α, β where $\alpha = x - y,\ \beta = x/y$.]

Problem 3 *In this question we write x_1, x_2, x_3 rather than x, y, z.*

Let $(x_1, x_2, x_3) \mapsto (x_1{}', x_2{}', x_3{}')$, where $x_i{}' = \sum_j \Lambda_{ij} x_j$, denote a rotation. This has the consequence that the matrix Λ is orthogonal (i.e. $\Lambda^T \Lambda = I$) so that $x_j = \sum_k \Lambda_{kj} x_k{}'$.

Show explicitly, by direct calculation, that Laplace's operator is invariant under rotations i.e. that

$$\frac{\partial^2 \phi}{\partial {x_1}^2} + \frac{\partial^2 \phi}{\partial {x_2}^2} + \frac{\partial^2 \phi}{\partial {x_3}^2} = \frac{\partial^2 \phi}{\partial {x_1{}'}^2} + \frac{\partial^2 \phi}{\partial {x_2{}'}^2} + \frac{\partial^2 \phi}{\partial {x_3{}'}^2},$$

where ϕ is any function posessing continuous second order partial derivatives.

Show also, by direct calculation, that

$$\left(\frac{\partial \phi}{\partial x_1}\right)^2 + \left(\frac{\partial \phi}{\partial x_2}\right)^2 + \left(\frac{\partial \phi}{\partial x_3}\right)^2$$

is rotationally invariant.

Problem 4 *Find the general solution of the PDE*

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = 0.$$

Problem 5 *Show that the PDE*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

isn't invariant under the Galilean transformation $(x, t) \mapsto (x', t')$, where $x' = x - Vt$, $t' = t$, and V is constant.

Problem 6 *Classify each of the following PDEs as hyperbolic, parabolic or elliptic and find the general solution in each case.*

$$3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0,$$

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In the case of the third equation give an example of a fourth degree real multinomial in x, y which satisfies the equation.

Problem 7 *(i) Suppose that* a, b, c *are real constants such that the quadratic equation*

$$a + 2b\lambda + c\lambda^2 = 0$$

has distinct real roots λ_1, λ_2. *Show that under the transformation* $(x, y) \mapsto (\xi, \eta)$, *where* $\xi = x + \lambda_1 y$, $\eta = x + \lambda_2 y$, *Euler's equation*

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0$$

becomes

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

Obtain the general solution of Euler's equation when a, b, c *satisfy the above conditions.*

(ii) Classify the following partial differential equations as elliptic, parabolic or hyperbolic, and find the general solution in each case:

$$\begin{aligned} 3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} &= 0 \\ 4\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ 2\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} &= 0 \end{aligned}$$

Find a tri-nomial $u(x, y)$ *which satisfies the third of these equations.*

Problem 8 *Write down d'Alembert's solution of the wave equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x \in \mathbf{R}, \ t \geq 0.$$

Find the solution which satisfies the initial conditions

$$u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = \frac{1}{1 + x^2}.$$

Problem 9 *For each of the following PDEs apply the method of separation of variables and find the ordinary differential equations into which the PDE separates in each case:*

$$x^2\frac{\partial^2 u}{\partial x^2} = y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} \tag{16.1}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{16.2}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{16.3}$$

This is the three-dimensional wave equation expressed in cylindrical polar coordinates (r, θ, z).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{16.4}$$

This is the two-dimensional version of the diffusion equation.

Problem 10 *Use the method of separation of variables to find solutions of the PDE*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = u.$$

Notice that this is a linear homogeneous equation and therefore the superposition principle can be applied. Use the Fourier method to show that the solution of the equation in the semi-infinite strip $0 \leq x \leq \pi,\ y \geq 1,\ (x, y) \neq (\pi, 1)$ *which satisfies the conditions* $u(0, y) = 0,\ u(\pi, y) = 0,\ u(x, 1) = x$, *is*

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \mathrm{e}^{(1+n^2)(1-y)} \sin nx.$$

Problem 11 *Use the method of separation of variables to find solutions of the diffusion equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Show, in particular, that $u(x, t) = ax + b$ *where* a, b *are constants is a solution.*

Use your results to show (formally) that the solution of the diffusion equation subject to the conditions

$$u(0, t) = 0,\ u(1, t) = 1\ \forall t > 0, \quad u(x, 0) = x(x + 1),\ 0 \leq x \leq 1$$

is

$$u(x, t) = x + \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, \mathrm{e}^{-n^2\pi^2 t} \tag{16.5}$$

where

$$B_n = \frac{2}{n\pi}(-1)^{(n+1)} + \frac{4}{n^3\pi^3}((-1)^n - 1).$$

Prove, using equation 16.5, that $\lim_{t\to\infty} u(x, t) = x$.

Problem 12 *Consider the wave equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\sigma^2}\frac{\partial^2 u}{\partial t^2}$$

inside the box $\{(x,y,z) : 0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c\}$. *Use the method of separation of variables and the Fourier method to find the solution which satisfies the following conditions:*

$$u(0,y,z,t) = 0,\ u(a,y,z,t) = 0,\ u(x,0,z,t) = 0,\ u(x,b,z,t) = 0,$$

$$u(x,y,0,t) = 0,\ u(x,y,c,t) = 0,$$

$$u(x,y,z,0) = f(x,y,z), \quad \frac{\partial u}{\partial t}(x,y,z,0) = 0.$$

Problem 13 *Find the Fourier series for* $\cosh ax,\ a \ne 0,\ -\pi \le x \le \pi$. *By putting* $x = 0$ *and then* $x = \pi$ *in turn obtain the formulae*

$$\frac{\pi}{\sinh a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n \frac{2a}{a^2+n^2},$$

$$\pi \coth a\pi = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2+n^2}.$$

Problem 14 *Obtain the Fourier series for* $\cosh ax,\ a \in \mathbf{R},\ a \ne 0,\ x \in [-\pi, \pi]$.

Hence show that

$$\frac{\pi}{\sinh a\pi} = \frac{1}{a} + \sum_{n=1}^{\infty}(-1)^n \frac{2a}{a^2+n^2}$$

and that

$$\frac{1}{\sinh z} - \frac{1}{z} = \sum_{n=1}^{\infty}(-1)^n \frac{2z}{z^2+n^2\pi^2}, \quad z \in \mathbf{R},\ z \ne 0.$$

Deduce, by integrating from 0 *to* x, *that*

$$\ln\Big(\frac{\tanh x/2}{x/2}\Big) = \sum_{n=1}^{m}(-1)^n \ln\Big(1 + \frac{x^2}{n^2\pi^2}\Big) + \int_0^x R_m(z)\,dz,\ \ x \ne 0,\ \text{where}$$

$$R_m(z) = \sum_{n=m+1}^{\infty}(-1)^n \frac{2z}{z^2+n^2\pi^2}.$$

Hence show that

$$\ln\Big(\frac{\tanh x/2}{x/2}\Big) = \sum_{n=1}^{\infty}(-1)^n \ln\Big(1 + \frac{x^2}{n^2\pi^2}\Big),\ \ x \ne 0.$$

(You may assume that

$$\int \frac{dz}{\sinh z} = \ln\tanh(z/2),\ \text{and that } |R_m(z)| \le 2|z|/(m\pi^2).)$$

Problem 15 *Show that* $u = f(x\cos\theta + y\sin\theta - ct)$ *represents a wave in two dimensions, moving without change of shape, the direction of propagation making an angle* θ *with the* x*-axis.*

Problem 16 *Use the method of separation of variables to find a class of solutions* $\{u_n(x,t),\ n = 1,2,3,\dots\}$ *of the PDE*

$$x^2\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

subject to the boundary conditions $u(a,t) = 0,\ u(2a,t) = 0,\ \forall t.$

Hint: *To solve the ODE*

$$x^2\frac{d^2X}{dx^2} + \lambda^2 X = 0$$

change the variable to θ, *where* $x = \mathrm{e}^{\theta}$.

Problem 17 *Show that the solution of the diffusion equation*

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa}\frac{\partial u}{\partial t}$$

subject to the conditions

$$u(0,t) = 0, \quad u(1,t) = 1, \quad \forall t > 0; u(x,0) = 0, \quad 0 \le x \le 1$$

is

$$u(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,\mathrm{e}^{(-n^2\pi^2\kappa t)}\sin n\pi x.$$

What happens as $t \to \infty$*? Offer a physical interpretation of this problem, if you can. (Look back to our original derivation of the diffusion equation and think in terms of the given boundary conditions.)*

Problem 18 *A function* $f : [0,\pi] \to \mathbf{R}$ *is defined by*

$$f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi \end{cases}$$

Show that

$$f(x) = \frac{4}{\pi}\Big(\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots\Big).$$

Hint: *Extend the definition of* f *to* $[-\pi,0]$ *by setting* $f(-x) = -f(x)$; *this makes* f *an odd function on* $[-\pi,\pi]$ *and all the Fourier coefficients with respect to the functions* $\{\cos nx\}$ *are zero.*

Deduce that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Problem 19 *Sometimes we have occasion to use Leibnitz's formula for the n^{th} derivative of a product. Verify by induction that, for any suitably differentiable functions f, g,*

$$D^n(fg) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} D^r f D^{n-r} g.$$

Problem 20 *In plane polar coordinates (r, θ) Laplaces's operator is given by*

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}.$$

(You should be able to derive this starting from $x = r\cos\theta,\ y = r\sin\theta$ using the chain rule)

Find solutions of Laplace's equation by writing $\phi(r, \theta) = R(r)\Theta(\theta)$ and employing the method of separation of variables. You can solve the resulting equation for $R(r)$ by trying $R(r) = Kr^l$. Use your results to solve the following problem.

Show that the solution $\phi(r, \theta)$ of Laplaces's equation in the semi-circular region $r < a$, $0 < \theta < \pi$, which vanishes on the line $\theta = 0$ and takes the constant value A on the line $\theta = \pi$ and on the curved boundary $r = a$, is

$$\phi(r, \theta) = \frac{A}{\pi}\left[\theta + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{\sin n\theta}{n}\right].$$

(Strictly speaking, the origin and the point $r = a,\ \theta = 0$ should be excluded)

Problem 21 *A function $u(r, t)$ $(r = (x^2 + y^2 + z^2)^{1/2})$ satisfies the 3-dimensional heat equation*

$$\nabla^2 v = \frac{\partial v}{\partial t}.$$

Assuming the expression for ∇^2 given in Appendix 1 show that

$$\frac{\partial^2}{\partial r^2}(ru) = \frac{\partial}{\partial t}(ru)$$

Use the method of separation of variables to show that this equation has solutions of the form $u = \tilde{u}$ given by

$$\tilde{u}(r, t) = \frac{(A\cos\lambda r + B\sin\lambda r)\,\mathrm{e}^{-\lambda^2 t} + Cr + D}{r},$$

where A, B, C, D are constants.

Find the solution $u(r, t)$ in the space $0 \le r \le a$ which satisfies the boundary conditions $u(r, 0) = 1,\ 0 < r < a,\ u(a, t) = 0,\ t > 0$.

Problem 22 *Starting from our definitions of the elementary functions prove some of the standard identities e.g.*

$$\cos^2 z + \sin^2 z = 1, \quad \cosh^2 z - \sinh^2 z = 1, \quad z \in \mathbf{C},$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \quad z_1, z_2 \in \mathbf{C}, \quad \sin iz = i \sinh z, \quad \cos iz = \cosh z.$$

Is it true that $|\sin z| \le 1, \quad |\cos z| \le 1, \quad \forall z \in \mathbf{C}$*?*

Problem 23 *Verify that* $u(x, y) = \sin x \cosh y$*, satisfies the two dimensional Laplace equation and find the analytic function of which it is the real part. (Use the Cauchy Riemann equations to calculate the imaginary part)*

Problem 24 *For* $z = x + iy \in \mathbf{C}$*, define* $f(z)$ *by*

$$f(z) = \frac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0, \quad f(0) = 0.$$

Show that

$$\frac{f(z) - f(0)}{z} \to 0$$

as $z \to 0$ *along any straight line through the origin, but prove that* f *is not differentiable at the origin by examining what happens as* $z \to 0$ *along the curve* $x = y^2$*.*

Problem 25 *Find the largest domain* D *on which the function* $f : D \to \mathbf{C}$ *is analytic, where*

$$f(z) = \frac{z(1 + z)}{z^4 + 1}, \quad z \in \mathbf{C}\,.$$

Problem 26 *Suppose that* f *is analytic on a domain* D *and that*

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy.$$

State the Cauchy-Riemann equations. Find the most general analytic function of which $x^3 - 3xy^2$ *is the imaginary part; express your answer in terms of* z*.*

Problem 27 *Evaluate* $\int \overline{z}\, dz$*, where the integral is taken in the positive sense round the triangle whose vertices are at* $z = 0,\ z = 2 + i,\ z = 1 + 2i$*.*

Problem 28 *Evaluate* $\int (\Re z + \overline{z})\, dz$*, where* $\Re$ *denotes real part and the integral is taken in the positive sense round the triangle whose vertices are at* $z = -1$*,*
$z = 1$*,* $z = i$*.*

Problem 29 *Determine the most general analytic function of which u is the real part, where*

$$u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2.$$

(Express your answer in terms of $z = x + iy$)

Problem 30 *Give a complex variable argument to show that*

$$a \ln r + b\theta + c + \sum_n (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta),$$

where (r,θ) are polar coordinates, satisfies Laplace's equation.

Problem 31 *Find all the solutions of the equation $\cos z = 2$.*

Problem 32 *Suppose that $f : \mathbf{C} \to \mathbf{C}$ is an analytic function and $\Re f(z)$ is constant, $\forall z \in \mathbf{C}$. Prove that f is constant. (Here $\Re$ denotes real part)*

Problem 33 *Solve Laplace's equation*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to the boundary conditions

$$\begin{aligned} u(r,\theta=0) &= 0, \quad u(r,\theta=\pi/4) = 0, \quad 0 \le r < a, \\ u(a,\theta) &= C, \quad 0 < \theta < \pi/4 \end{aligned}$$

Problem 34 *Evaluate $\int_\gamma \overline{z}\, dz$, where $\gamma = \gamma_1 + \gamma_2$, γ_1 being the line segment from 1 to 0 and γ_2 being the line segment from 0 to $2+2i$.*

Problem 35 *Evaluate $\int_\gamma \mathrm{e}^z\, dz$, where γ is any smooth curve connecting $2i$ to $1+i$ expressing your answer in the form $X + iY$.*

Problem 36 *Use Cauchy's integral formula to evaluate*

$$\int_\gamma \frac{\mathrm{e}^{z^2}}{z-1}\, dz,$$

where γ is the rectangle bounded by $x = 0,\ x = 3,\ y = -1,\ y = 1$ and deduce that

$$-\int_{-1}^{1} \frac{y\,\mathrm{e}^{-y^2}\, dy}{y^2+1} + \int_{-1}^{1} \frac{y\,\mathrm{e}^{9-y^2}}{y^2+4}(y\cos 6y - 2\sin 6y)\, dy = 0$$

together with another equation which you are asked to find.

(Write out the integral explicitly and extract the real and imaginary parts)

Problem 37 *Use Laurent's theorem to show that*

$$\exp\Big(\frac{x}{2}(z+1/z)\Big) = \sum_{n=-\infty}^{\infty} K_n(x)z^n \quad \forall z \in \mathbf{C}-\{0\}$$

Find an integral representation for the functions $K_n(x)$ and try to reproduce the analysis of example 6.3 of Chapter 6 (in relation to the Bessel functions) for the $K_n(x)$.

Problem 38 *Show by an application of Laurent's theorem to the function $\mathrm{e}^{\frac{x}{2}(z+1/z)}$ that*

$$\mathrm{e}^{\frac{x}{2}(z+1/z)} = \sum_{n=-\infty}^{\infty} K_n(x)z^n, \ \forall z \in \mathbf{C}-\{0\} \tag{16.6}$$

where

$$K_n(x) = \frac{1}{\pi}\int_0^{\pi} \mathrm{e}^{x\cos\theta}\cos n\theta \, d\theta.$$

(Parametrise the unit circle, centre $z=0$, by $\gamma(\theta)=\mathrm{e}^{i\theta}, \ -\pi \le \theta \le \pi$.)

Show, by differentiating equation 16.6 with respect to x and z, in turn, that

$$\begin{aligned} xK_n'(x) + nK_n(x) &= xK_{n-1}(x) \\ xK_n'(x) - nK_n(x) &= xK_{n+1}(x) \end{aligned}$$

Deduce that $K_n(x)$ satisfies the differential equation

$$x^2K_n''(x) + xK_n'(x) - (x^2+n^2)K_n(x) = 0.$$

Problem 39 *Evaluate*

$$\int_C \frac{\mathrm{e}^z}{z(z-3)}\,dz$$

where C is

(a) the circle $|z-3|=1$, (b) the circle $|z-i|=1/4$.

Problem 40 *Use Cauchy's residue theorem to evaluate*

$$\int_\gamma \frac{z\,\mathrm{e}^{iz}\,dz}{z^2+1},$$

where γ is the contour which consists of the portion of the x-axis which lies between $-R$ and R together with the semi-circle γ_R, where
$\gamma_R(\theta) = Re^{i\theta}, \ 0 \le \theta \le \pi, \ R > 1$.

Deduce that

$$\int_0^\infty \frac{x\sin x\,dx}{x^2+1} = \frac{\pi}{2\,\mathrm{e}}.$$

(You should prove that

$$\int_{\gamma_R} \frac{z\,\mathrm{e}^{iz}\,dz}{z^2+1} \to 0, \ \text{as } R\to\infty.)$$

Problem 41 *State the value of* $\int_\gamma \mathrm{e}^{iz^2}\, dz$*, where* γ *is the contour shown in the diagram.*

Prove that

$$\int_{\gamma_R} \mathrm{e}^{iz^2}\, dz \to 0, \text{ as } R \to \infty,$$

where $\gamma_R(\theta) = R\,\mathrm{e}^{i\theta},\ 0 \le \theta \le \pi/4.$

(You may assume that $\sin\theta \ge 2\theta/\pi,\ 0 \le \theta \le \pi/2.$*)*

Deduce that

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(You may assume that $\int_0^\infty \mathrm{e}^{-r^2}\, dr = \sqrt{\pi}/2.$*)*

Problem 42 *Use Cauchy's residue theorem to evaluate*

$$\int_\gamma \frac{\mathrm{e}^{iaz}}{1+z^2}\, dz \quad (a > 0),$$

where γ *is the contour which consists of the portion of the* $x-$*axis which lies between* $-R$ *and* R *together with the semi-circle* γ_R*, where*
$\gamma_R(\theta) = R\mathrm{e}^{i\theta},\ 0 \le \theta \le \pi,\ R > 1$*. Deduce that*

$$\int_0^\infty \frac{\cos ax}{1+x^2}\, dx = \frac{\pi}{2}\mathrm{e}^{-a}.$$

(You may assume that $\int_{\gamma_R} \frac{\mathrm{e}^{iaz}}{1+z^2}\, dz \to 0$ *as* $R \to \infty$*.)*

Problem 43 *Let S denote the rectangle whose sides are $x = \pm R$, $y = 0$, $y = 1$. Find the poles of the function f given by*

$$f(z) = \frac{e^{iaz}}{\cosh \pi z} \quad (a > 0)$$

and show that the only pole of f which lies inside S is at $z = i/2$. Show that the residue of f at the pole $z = i/2$ is $-i(e^{-a/2})/\pi$.

Prove, by integrating f round the rectangle S, that

$$\int_0^\infty \frac{\cos ax}{\cosh \pi x}\, dx = \frac{1}{2}\mathrm{sech}\,(a/2).$$

(The integral of f along either of the sides of S which are parallel to the imaginary axis tends to 0 as $R \to \infty$ and your proof should include a demonstration that this is the case, for one of these sides.)

Problem 44 *By integrating*

$$f(z) = \frac{\mathrm{Log}\,(1 - iz)}{(z^2 + 1)^2}$$

round contour which consists of the portion of the real axis between $-R$ and R, together with the semi-circle γ_R given by $\gamma_R(\theta) = R\,e^{i\theta},\ 0 \le \theta \le \pi$ show that

$$\int_0^\infty \frac{\ln(1 + x^2)\, dx}{(x^2 + 1)^2} = \frac{\pi}{4}(2 \ln 2 - 1).$$

(Log denotes the principal value of the logarithm)

Chapter 17

Examination Questions 1998 — 2002

17.1 CM211A Examination Questions — June 1998

1. (i) Classify the following PDEs as elliptic, parabolic or hyperbolic, and write down the general solution in each case:

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$5\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

Find a real-valued fourth order multinomial solution of the second of these equations.

(ii) Obtain the condition for the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

to be invariant under the *non-singular* linear transformation $(x, y) \mapsto (\xi, \eta)$ given by

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \zeta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (\alpha, \beta, \gamma, \zeta \in \mathbf{R}).$$

and deduce that if $\zeta = 0$ the transformation matrices which leave the PDE invariant are scalar multiples of the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

2. Throughout this question r denotes the distance from an origin in three-dimensional space.

Heat conducting material occupies the space $0 \le r \le a$ and at time t the temperature $u(r,t)$ satisfies the PDE

$$\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{\kappa}\frac{\partial}{\partial t}(ru),$$

where κ is a positive constant. Use the method of separation of variables to show that the equation has solutions of the form

$$u(r,t) = \frac{a + br + \sum_\lambda (C_\lambda \cos\lambda r + D_\lambda \sin\lambda r)e^{-\lambda^2\kappa t}}{r}.$$

Use this class of solutions to find the solution of the given PDE which satisfies the following conditions:

(a) u is finite in the space $0 \le r \le a$

(b) $u(r,0) = 1 + r,\ 0 \le r \le a$

(c) $u(a,t) = 1,\ \forall t > 0.$

Use your solution to show *carefully* that

$$\lim_{t\to\infty} u(r,t) = 1,\ 0 \le r \le a$$

3. Throughout this question (r,θ) denote plane polar coordinates.

Show by a complex variable argument, or otherwise, that the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

has solutions of the form

$$u(r,\theta) = c_0 + d_0 \ln r + \sum_n (a_n r^n + b_n r^{-n})\cos n\theta + \sum_n (f_n r^n + g_n r^{-n})\sin n\theta,$$

where n is an integer.

Use this result to find the solution of Laplace's equation in the space $a \le r \le b$ which satisfies the boundary conditions

$$u(a,\theta) = 1,\ u(b,\theta) = \cos^4\theta,\ 0 \le \theta \le 2\pi.$$

4. This question was on Legendre polynomials which are no longer part of the course.

5. (i) Let $f : D \to \mathbf{C}$, where D is a domain. Explain what is meant by the statements:

(a) f is *differentiable* at $z_0 \in D$,

(b) f is *analytic* at $z_0 \in D$.

Use your definition to investigate the differentiability of $f : \mathbf{C} \to \mathbf{C}$,
where $f(z) = z|z|^2$.

(ii) Find the most general analytic function whose imaginary part is given by
$v(x, y) = \cos x \sinh y$.

(iii) Evaluate

$$\int_{\gamma_1} |z|^2 \, dz,$$

where γ_1 is the straight line connecting $z = i$ to $z = 1 - 2i$.

(iv) Evaluate

$$\int_{\gamma_2} \frac{dz}{z},$$

where γ_2 is the straight line connecting $z = i$ to $z = 1 - \sqrt{3}i$.

6. (i) Write down the Taylor expansion of $e^z, z \in \mathbf{C}$ and hence evaluate the residue of the function e^z/z^n, where n is a positive integer, at $z = 0$.

Use Cauchy's residue theorem to evaluate

$$\int_\gamma \frac{e^z}{z^n} \, dz,$$

where γ is the unit circle parametrised by $\gamma(\theta) = e^{i\theta},\ 0 \le \theta \le 2\pi$.

Deduce that

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - (n-1)\theta) \, d\theta = \frac{2\pi}{(n-1)!}\,.$$

(ii) Prove, by integrating

$$\int \frac{\mathrm{Log}(1 - iz)}{z^2 + 1} \, dz,$$

round the contour indicated in the diagram, that

$$\int_{-\infty}^{\infty} \frac{\ln(1 + x^2)}{1 + x^2} \, dx = 2\pi \ln 2.$$

(Log denotes the principal value of the (complex) logarithm and ln denotes the usual, real-valued logarithm)

17.2 Solutions

Some of the solutions are given in greater detail than others; in order to benefit you should make a serious attempt to do the questions yourself before looking at the solutions.

1. (i) The PDE

$$\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y^2}=0$$

has auxiliary equation $1+4\lambda+\lambda^2=0$ with solutions $\lambda=\lambda_1=-2+\sqrt{3},\ \lambda=\lambda_2=-2-\sqrt{3}$. The PDE is therefore hyperbolic with GS $u=f_1(x+\lambda_1 y)+f_2(x+\lambda_2 y)$, where f_1, f_2 are arbitrary $\mathbf{C}^2$ functions.

Similarly, the PDE

$$5\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 u}{\partial y^2}=0$$

has auxiliary equation $5-4\lambda+\lambda^2=0$ so that $\lambda=\lambda_1=2+i,\ \lambda=\lambda_2=2-i$. The PDE is elliptic with general solution $u=g_1(x+\lambda_1 y)+g_2(x+\lambda_2 y)$, where g_1, g_2 are arbitrary $\mathbf{C}^2$ functions. Taking $g_2=0$ and $g_1(z)=z^4$ we see that the complex function $[x+(2+i)y]^4$ is a solution. Expanding this out using $(2+i)^2=3+4i,\ (2+i)^3=2+11i,\ (2+i)^4=-7+24i$ and extracting the real part we find that $u(x,y)=x^4+8x^3y+18x^2y^2+8xy^3-7y^4$ is a solution of the required type. [Equally well, the imaginary part of $[x+(2+i)y]^4$ provides a fourth order multinomial solution of the PDE]

(ii) Under the change of variable $\xi=\alpha x+\beta y,\ \eta=\gamma x+\zeta y$ we find, using the chain rule

$$\frac{\partial u}{\partial x}=\alpha\frac{\partial u}{\partial \xi}+\gamma\frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial y}=\beta\frac{\partial u}{\partial \xi}+\zeta\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2}=\alpha^2\frac{\partial^2 u}{\partial \xi^2}+\gamma^2\frac{\partial^2 u}{\partial \eta^2}+2\alpha\gamma\frac{\partial^2 u}{\partial \xi\,\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2}=\beta^2\frac{\partial^2 u}{\partial \xi^2}+\zeta^2\frac{\partial^2 u}{\partial \eta^2}+2\beta\zeta\frac{\partial^2 u}{\partial \xi\,\partial \eta}$$

$$\frac{\partial^2 u}{\partial x\,\partial y}=\alpha\beta\frac{\partial^2 u}{\partial \xi^2}+\gamma\zeta\frac{\partial^2 u}{\partial \eta^2}+(\alpha\zeta+\gamma\beta)\frac{\partial^2 u}{\partial \xi\,\partial \eta}$$

Substituting in the given PDE we find that

$$\frac{\partial^2 u}{\partial \xi^2}[\alpha^2+\alpha\beta+\beta^2]+\frac{\partial^2 u}{\partial \xi\,\partial \eta}[2\alpha\gamma+2\beta\zeta+\alpha\zeta+\gamma\beta]+\frac{\partial^2 u}{\partial \eta^2}[\gamma^2+\zeta^2+\gamma\zeta]=0$$

and the given PDE is invariant under the transformation in question provided

$$2\alpha\gamma+2\beta\zeta+\alpha\zeta+\gamma\beta=\alpha^2+\alpha\beta+\beta^2$$

$$\gamma^2 + \zeta^2 + \gamma\zeta = \alpha^2 + \alpha\beta + \beta^2$$

If $\zeta = 0$ we immediately obtain $\gamma = 2\alpha + \beta$ (γ cannot be zero since we're given that $\alpha\zeta - \beta\gamma \neq 0$) and substitution in the second of the two constraints yields $\alpha = 0$ or $\alpha = -\beta$. If $\alpha = 0$ we immediately obtain $\gamma = \beta$ and we get

$$T = \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

whereas $\alpha = -\beta$ yields $\gamma = -2\beta + \beta = -\beta$ and

$$T = \beta \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

as stated in the question.

2. We're given the PDE

$$\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{\kappa}\frac{\partial}{\partial t}(ru)$$

Putting $ru = R(r)T(t)$ gives (by separation of variables) the equations

$$\frac{1}{R}\frac{d^2R}{dr^2} = \frac{1}{\kappa}\frac{dT}{dt} = -\lambda^2,$$

$$\frac{d^2R}{dr^2} + \lambda^2 R + 0, \quad \frac{dT}{dt} + \lambda^2\kappa T = 0,$$

where λ is a parameter. If $\lambda = 0$ we get $R = (A' + B'r),\ T = \text{constant}$ and if $\lambda \neq 0$

$$R = A_\lambda \cos\lambda r + B_\lambda \sin\lambda r, \quad T = E_\lambda e^{-\lambda^2\kappa t}$$

in an obvious notation. Since the PDE we're dealing with is linear and homogeneous any linear combination of such solutions is a solution. We therefore obtain a class of solutions of the form

$$ru = A + Br + \sum_{\lambda\in\Lambda}(C_\lambda \cos\lambda r + D_\lambda \sin\lambda r)e^{-\lambda^2\kappa t}$$

as required.

As regards the given boundary value problem, the fact that u is to be finite in the space $0 < r \leq a$ demands that $A = 0,\ C_\lambda = 0$ leading to the class of solutions

$$u(r,t) = B + \sum_{\lambda\in\Lambda} D_\lambda \frac{\sin\lambda r}{r} e^{-\lambda^2\kappa t}$$

Next, the condition $u(a,t) = 1,\ \forall t > 0$ demands that

$$1 = B + \sum_{\lambda\in\Lambda} D_\lambda \frac{\sin\lambda a}{a} e^{-\lambda^2\kappa t},\ \forall t > 0.$$

To satisfy this we must have $B = 1$ and $\lambda a = n\pi,\ (n = 1, 2, \ldots)$ — ignoring the trivial possibility $D_\lambda = 0$. We're therefore led to consider the class of solutions given by

$$u(r,t) = 1 + \sum_{n=1}^{\infty} D_n \frac{\sin\left(\frac{n\pi r}{a}\right)}{r} e^{-n^2\pi^2\kappa t/a^2}, \quad 0 < r \le a.$$

The requirement $u(r, 0) = 1 + r$ gives

$$1 + r = 1 + \sum_{n=1}^{\infty} \frac{D_n}{r} \sin\left(\frac{n\pi r}{a}\right), \quad 0 \le r \le a.$$

The method of Fourier Series now gives

$$D_n \frac{a}{2} = \int_0^a r^2 \sin\left(\frac{n\pi r}{a}\right) dr$$

and a straightforward integration by parts gives

$$D_n = \frac{2a^2(-1)^{n+1}}{n\pi} + \frac{4a^2}{n^3\pi^3}\Big[(-1)^n - 1\Big].$$

We observe that

$$\left|\frac{\sin\left(\frac{n\pi r}{a}\right)}{r}\right| = \left|\left(\frac{n\pi}{a}\right)\frac{\sin(n\pi r/a)}{n\pi r/a}\right| \le \frac{n\pi}{a} \quad \text{[Why?]}$$

It follows that

$$\left|D_n \frac{\sin(n\pi r/a)}{r}\right| \le K, \quad 0 < r \le a$$

for some constant K, and for all n. [Fill in the details]

We can write our solution as $u(r, t) = 1 + I_1 + I_2$ where

$$I_1 = \sum_{n=1}^{m} D_n \frac{\sin(n\pi r/a)}{r} e^{-n^2\pi^2\kappa t/a^2}, \quad I_2 = \sum_{n=m+1}^{\infty} D_n \frac{\sin(n\pi r/a)}{r} e^{-n^2\pi^2\kappa t/a^2}$$

Obviously $I_1 \to 0$ as $t \to \infty$ since I_1 is a finite sum of terms, each of which tends to zero as t tends to infinity (property of the negative exponential). As regards I_2 we can note that

$$|I_2| \le \sum_{n=m+1}^{\infty} K e^{-n\pi^2\kappa t/a^2} = K e^{-(m+1)\pi^2\kappa t/a^2}/(1 - e^{-\pi^2\kappa t/a^2}) \to 0$$

as $t \to \infty$.

Here we've use the fact that $e^{-n^2\pi^2\kappa t/a^2} < e^{-n\pi^2\kappa t/a^2}$ for all $n > 1$ — if you draw the graph of the negative exponential this will be clear; also note that $\sum_{n=m+1}^{\infty} e^{-n\pi^2\kappa t/a^2}$ is a geometric series with common ratio equal to $e^{-\pi^2\kappa t/a^2}$. Fill in the details.

We conclude that both I_1 and I_2 tend to zero as t tends to infinity. It follows that

$$\lim_{t\to\infty} u(r,t) = 1, \quad 0 < r \le a.$$

3. We can argue that $\log z$, z^n are analytic, the first in an appropriate cut plane, the second everywhere — except at $z = 0$ in the case when n is a negative integer. The real and imaginary parts of these functions therefore satisfy the two-dimensional Laplace equation. Thus, $\ln r,\ r^n \cos n\theta,\ r^{-n}\cos n\theta,\ r^n \sin n\theta,\ r^{-n}\sin n\theta,\ (n = 0, 1, 2, \ldots)$ satisfy Laplace's equation. Laplace's equation is linear and homogeneous, so any linear combination of solutions is also a solution. We see that

$$u(r,\theta) = c_0 + d_0 \ln r + \sum_n (a_n r^n + b_n r^{-n}) \cos n\theta + \sum_n (f_n r^n + g_n r^{-n}) \sin n\theta$$

(in an obvious notation) will provide a solution of the two-dimensional Laplace equation.

A standard calculation shows that (do it!)

$$\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

The conditions $u(a,\theta) = 1,\ u(b,\theta) = \cos^4\theta,\ 0 \le \theta \le 2\pi$ demand that

$$1 = c_0 + d_0 \ln a + \sum_n (a_n a^n + b_n a^{-n}) \cos n\theta + \sum_n (f_n a^n + g_n a^{-n}) \sin n\theta,\ \forall\theta \in [0, 2\pi]$$

and

$$\frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta = c_0+d_0 \ln b+\sum_n (a_n b^n+b_n b^{-n}) \cos n\theta+\sum_n (f_n b^n+g_n b^{-n}) \sin n\theta,\ \forall\theta \in [0, 2\pi]$$

A moment's consideration shows that in order to satisfy these equations for all $\theta \in [0, 2\pi]$ we need $f_n,\ g_n$ to be zero for all n, $c_0 + d_0 \ln a = 1,\ b_n = -a_n a^{2n}$ (for all n) together with

$$\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta = c_0 + d_0 \ln b + \sum_n a_n (b^n - a^{2n} b^{-n}) \cos n\theta,\ \forall\theta \in [0, 2\pi]$$

We conclude that

$$\frac{3}{8} = c_0 + d_0 \ln b,\ a_2(b^2 - a^4/b^2) = \frac{1}{2},\ a_4(b^4 - a^8/b^4) = \frac{1}{8}$$

with all the other a_n equal to zero. Solving for $c_0,\ d_0$ we see that the unique solution (recall the uniqueness theorem for Laplace's equation) of the given boundary value problem is

$$u(r,\theta) = 1 - \frac{5\ln a}{8\ln(a/b)} + \frac{5\ln r}{8\ln(a/b)} + \left(r^2 - \frac{a^4}{r^2}\right)\frac{b^2}{2(b^4 - a^4)}\cos 2\theta + \left(r^4 - \frac{a^8}{r^4}\right)\frac{b^4}{8(b^8 - a^8)}\cos 4\theta,$$

$a \le r \le b,\ 0 \le \theta < 2\pi$.

4. We have omitted this question on Legendre polynomials which are no longer part of this course.

5. (i) (a) f differentiable at $z_0 \in D$ implies that $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists; in ϵ, δ language this means that given any $\epsilon > 0\ \ \exists \delta > 0$ such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - L\right| < \epsilon,$$

for some $L \in \mathbf{C}$ and all z such that $0 < |z - z_0| < \delta$.

(b) The statement that f is analytic at $z_0 \in D$ means that $\exists r > 0$ such that f is differentiable at each point of $N_r(z_0)$, the open disc centre z_0 and radius r.

Note that for any complex number z, $|z|^2 = z\overline{z}$, where $\overline{z}$ denotes the complex conjugate of z. For $f(z) = z|z|^2$ we have

$$\frac{f(z+w)-f(z)}{w} = \frac{(z+w)^2(\overline{z}+\overline{w}) - z^2\overline{z}}{w} = 2|z|^2 + w\overline{z} + z^2\frac{\overline{w}}{w} + 2\overline{w}z + |w|^2 \qquad (17.1)$$

If $z = 0$ this expression tends to zero as $w \to 0$ so that $f'(0)$ exists and equals zero but if $z \neq 0$ all the terms in (17.1) have well defined limits as $w \to 0$ *except* the term $z^2\overline{w}/w$ which does *not* have a limit as $w \to 0$. We can see this as follows. For $w \in \mathbf{R}$ $\ \overline{w}/w = 1$, whilst for $w = i\lambda, \lambda \in \mathbf{R}$ $\overline{w}/w = -1$. It follows that $\lim_{w\to 0} \overline{w}/w$ does not exist. In conclusion, f is differentiable at $z = 0$, but at no other point.

(ii) The Cauchy-Riemann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

With $v(x, y) = \cos x \sinh y$ the first CR equation gives

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \quad u(x, y) = \sin x \cosh y + f(y).$$

Plugging this expression in the second CR equation gives

$$-\sin x \sinh y = -\sin x \sinh y - f'(y), \quad f(y) = C,\ C \in \mathbf{R}\,.$$

The required analytic function is therefore

$$f(z) = \sin x \cosh y + i\cos x \sinh y + C = \sin x \cos iy + \cos x \sin iy + C = \sin(x+iy) + C \equiv \sin z + C.$$

(iii) We can parametrise γ_1 by

$$\gamma_1(t) = t(1 - 2i - i) + i = i + t(1 - 3i),\ 0 \le t \le 1.$$

We then obtain

$$\int_{\gamma_1} |z|^2\, dz = (1-3i)\int_0^1 (t^2 + (1-3t)^2)\, dt = \frac{4}{3}(1-3i).$$

[Check through the details]

(iv) Log$z$ is a primitive for $1/z$ in the region in question so

$$\int_{\gamma_2} \frac{dz}{z} = \Big[\mathrm{Log} z\Big]_i^{1-\sqrt{3}i}$$

$= \ln|1-\sqrt{3}i| + i\mathrm{Arg}(1-\sqrt{3}i) - \ln|i| - i\mathrm{Arg} i = \ln 2 + i(-\pi/3) - i(\pi/2) = \ln 2 - (5\pi i/6)$

6. (i) The Taylor exapnsion of e^z is given by

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^{n-1}}{(n-1)!} + \frac{z^n}{n!} + \cdots \quad \forall z \in \mathbf{C}.$$

We then have

$$\frac{e^z}{z^n} = \frac{1}{z^n} + \frac{1}{z^{n-1}} + \cdots + \frac{1}{(n-1)!}\frac{1}{z} + \frac{1}{n!} + \frac{z}{(n+1)!} + \cdots$$

(This is the Laurent expansion of e^z/z^n about $z=0$ where e^z/z^n has an n-th order pole; the residue is the coefficient of $1/z$ i.e. the residue is $1/(n-1)!$ By Cauchy's residue theorem

$$\int_\gamma \frac{e^z}{z^n}\, dz = \frac{2\pi i}{(n-1)!}$$

where γ is the unit circle parametrised by $z(\theta) = \gamma(\theta) = e^{i\theta}, \quad 0 \le \theta < 2\pi$. We conclude that

$$\int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{ni\theta}} ie^{i\theta}\, d\theta = \frac{2\pi i}{(n-1)!}$$

Dividing out the i on both sides and extracting the real part gives

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - (n-1)\theta)\, d\theta = \frac{2\pi}{(n-1)!}, \quad n = 1, 2, 3, \ldots$$

(ii) The function Log$(1-iz)$ has a branch point at $z=-i$. It follows that

$$\frac{\mathrm{Log}(1-iz)}{z^2+1}$$

is analytic in the domain Im$z > -1$ except for a *simple* pole at $z=i$. The residue at $z=i$ is $\ln 2/(2i)$ [check this, remembering that we are dealing with the principal value of the logarithm] From Cauchy's residue theorem, applied to the contour which consists of the portion of the real-axis given by $-R \le x \le R$ and the semi-circle C_R parametrised by $z(\theta) = Re^{i\theta},\ 0 \le \theta \le \pi$, we obtain

$$\int_{-R}^{R} \frac{\operatorname{Log}(1-ix)}{x^2+1}\,dx + \int_{C_R} \frac{\operatorname{Log}(1-iz)}{z^2+1}\,dz = (2\pi i)\frac{\ln 2}{2i} = \pi \ln 2.$$

Now,

$$\left|\int_{C_R}\right| = \left|\int_0^{\pi} \frac{\operatorname{Log}(1-iRe^{i\theta})iRe^{i\theta}\,d\theta}{R^2e^{2i\theta}+1}\right| \le \int_0^{\pi} \frac{|\operatorname{Log}(1-iRe^{i\theta})|R\,d\theta}{|R^2e^{2i\theta}+1|}.$$

But

$$|R^2e^{2i\theta}+1| \ge ||R^2e^{2i\theta}|-1| = R^2-1$$

and therefore

$$\left|\int_{C_R}\right| \le \frac{R}{R^2-1}\int_0^{\pi} |\operatorname{Log}(1-iRe^{i\theta})|\,d\theta.$$

Since $|1-iRe^{i\theta}| \le 1+R$ (by the triangle inequality) and $|\operatorname{Arg} z| \le \pi$ it follows that

$$|\operatorname{Log}(1-iRe^{i\theta})| = \sqrt{(\ln|1-iRe^{i\theta}|)^2 + (\operatorname{Arg}(1-iRe^{i\theta}))^2} \le \sqrt{(\ln(1+R))^2+\pi^2} \le \sqrt{2}\ln(1+R)$$

for all large R. Letting $R \to \infty$ we see that $\int_{C_R} \to 0$ because $(\ln R)/R \to 0$ as $R \to \infty$.

We conclude that

$$\int_{-\infty}^{\infty} \frac{\ln\sqrt{1+x^2}}{1+x^2}\,dx = \pi\ln 2, \quad \int_{-\infty}^{\infty} \frac{\ln(1+x^2)}{1+x^2}\,dx = 2\pi\ln 2$$

because $\operatorname{Log}(1-ix) = \ln\sqrt{1+x^2} - i\arctan x = \frac{1}{2}\ln(1+x^2) - i\arctan x$.

It's perhaps worth noting that this integral can be computed by elementary methods. If you'd like to take up the challenge, start by making the substitution $x = \tan\theta$. The resulting integral can be evaluated by an elementary but rather cunning method described in an old book — Integration, by R.P. Gillespie, originally published by Oliver and Boyd in their series of Mathematical Texts, page 24.

17.3 CM211A Examination Questions — June 1999

1. Consider the partial differential equation (PDE)

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2} = 0 \qquad (17.2)$$

where a, b, c, are non-zero real constants such that the quadratic equation

$$a + 2b\lambda + c\lambda^2 = 0$$

has distinct roots $\lambda = \lambda_1,\ \lambda = \lambda_2$. Find the general solution of the PDE (17.2) by making the change of variable $(x, y) \mapsto (\xi, \eta)$, where $\xi = x + \lambda_1 y,\ \eta = x + \lambda_2 y$. Use your result to write down the general solution of the PDE

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0,\ x \in \mathbf{R},\ y \geq 0 \tag{17.3}$$

Hence show that the solution of the PDE (17.3) which satisfies the conditions

$$u(x, 0) = x, \quad \frac{\partial u}{\partial y}(x, 0) = x, \quad (x \in \mathbf{R})$$

is

$$u(x, y) = x + xy - 2y^2/3.$$

2. (i) Show, by the method of separation of variables, that the partial differential equation (PDE)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{17.4}$$

possesses solutions of the form

$$\widetilde{u}(\omega, x, t) = (A_\omega \cos \omega x + B_\omega \sin \omega x)(C_\omega \cos \omega t + D_\omega \sin \omega t),$$

where ω is a real parameter.

(ii) Use the result of part (i) to solve equation (17.4) subject to the conditions

$$u(0, t) = u(2, t) = 0 \ \ \forall t \geq 0,$$

$$\frac{\partial u}{\partial t}(x, 0) = 0,\ u(x, 0) = \sin \pi x, \quad 0 \leq x \leq 2.$$

(iii) A string of natural length equal to 4 has its ends fixed at $x = 0$ and $x = 4$. At time $t = 0$ it is drawn aside through a small distance h at the point $x = 1$ and immediately released from rest. In the subsequent motion the transverse displacement of the string satisfies the wave equation (17.4). Show that

$$u(x, t) = \frac{32h}{3\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/4)}{n^2} \sin(n\pi x/4) \cos(n\pi t/4).$$

3. We have omitted this question on Legendre polynomials which are no longer part of this course.

4. (i) By noting that z^n ($z = x + iy$, $n = 0, 1, 2, 3, \ldots$) are analytic functions, or otherwise, show that the solution of the 2-dimensional Laplace equation (expressed in plane polar coordinates (r, θ))

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

which satisfies the boundary conditions

$$u(r, \theta = 0) = 0,\ u(r, \theta = \pi/6) = 0,\ 0 \le r < a, \quad u(a, \theta) = C,\ 0 < \theta < \pi/6$$

is

$$u(r, \theta) = \frac{4\pi}{C}\sum_{k=0}^{\infty}\left(\frac{r}{a}\right)^{6(2k+1)}\frac{\sin 6(2k+1)\theta}{(2k+1)}.$$

(ii) Verify that the functions

$$u_n(x, y) = \sinh(n\pi x)\sin(n\pi y), \quad (n = 1, 2, 3, \ldots)$$

satisfy the 2-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence show that the solution of Laplace's equation in the space $0 < x < 1,\ 0 < y < 1$ which satisfies the boundary conditions

$$u(x, 0) = 0, \quad 0 \le x < 1, \quad u(0, y) = 0, \quad 0 \le y \le 1,$$
$$u(x, 1) = 0, \quad 0 \le x < 1, \quad u(1, y) = 1, \quad 0 < y < 1,$$

is

$$u(x, y) = 4\sum_{r=0}^{\infty}\frac{\sinh[(2r+1)\pi x]}{\sinh[(2r+1)\pi]}\frac{\sin[(2r+1)\pi y]}{(2r+1)}.$$

5. (i) Let $f : D \to \mathbf{C}$, where D is a domain. Explain what is meant by the statements:

(a) f is *differentiable* at $z_0 \in D$

(b) f is *analytic* at $z_0 \in D$.

(ii) Find the most general analytic function whose real part is given by $u(x, y) = \cosh x \cos y$.

(iii) Evaluate $\int \overline{z}\,\mathrm{Re} z\, dz$ along the following paths connecting $z = 0$ to $z = 1 + i$:

(a) The straight line from $z = 0$ to $z = 1 + i$,

(b) The straight line from $z = 0$ to $z = 1$, followed by the straight line from $z = 1$ to $z = 1 + i$.

Does $\overline{z}\,\mathrm{Re} z$ possess a primitive in $\mathbf{C}$? Explain your reasoning.

(iv) Evaluate $\int_\gamma \mathrm{Log}\, z\, dz$, where γ is the semi-circular path, in the *positive* i.e. the anti-clockwise sense, from $A(z = 1)$ to $B(z = i)$ with AB as diameter. Express your answer in the form $X + iY$.

(Log denotes the *principal* value of the logarithm)

6. (i) Let $\Omega = \{r : r > 0,\ r \neq \sqrt{2},\ r \neq \sqrt{5}\}$. For $r \in \Omega$ let γ_r denote the circle centre i and radius r and set

$$I_r = \int_{\gamma_r} \frac{e^{-z^2}\,dz}{(z-1)(z-2)}.$$

Evaluate I_r for all $r \in \Omega$.

(ii) By considering

$$\int_{C_R} \frac{e^{iz}\,dz}{(1+z^2)^2},$$

where C_R is the contour which consists of the portion of the real axis from $z = -R$ to $z = R$, together with the semi-circle γ_R given by $\gamma_R(\theta) = Re^{i\theta}$,
$0 \leq \theta \leq \pi$, prove that

$$\int_0^\infty \frac{\cos x\,dx}{(1+x^2)^2} = \frac{\pi}{2e}.$$

17.4 Solutions

1. Application of the chain rule to the given transformation yields

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta = \left(\frac{\partial}{\partial\,\xi} + \frac{\partial}{\partial\,\eta}\right)u$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = \lambda_1 u_\xi + \lambda_2 u_\eta = \left(\lambda_1\frac{\partial}{\partial\,\xi} + \lambda_2\frac{\partial}{\partial\,\eta}\right)u$$

Calculating the second derivatives in the usual way we find that Euler's equation becomes

$$a[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}] + 2b[\lambda_1 u_{\xi\xi} + (\lambda_1+\lambda_2)u_{\xi\eta} + \lambda_2 u_{\eta\eta}] + c[{\lambda_1}^2 u_{\xi\xi} + 2\lambda_1\lambda_2 u_{\xi\eta} + {\lambda_2}^2 u_{\eta\eta}] = 0$$

so that

$$u_{\xi\xi}[a + 2b\lambda_1 + c{\lambda_1}^2] + u_{\eta\eta}[a + 2b\lambda_2 + c{\lambda_2}^2] + u_{\xi\eta}[2b(\lambda_1+\lambda_2) + 2c\lambda_1\lambda_2 + 2a] = 0$$

Choosing λ_1, λ_2 as the roots of the quadratic $a + 2b\lambda + c\lambda^2 = 0$ we derive, using
$\lambda_1 + \lambda_2 = -(2b)/c, \quad \lambda_1\lambda_2 = a/c$,

$$u_{\xi\eta}\left[-\frac{2b^2}{c} + a + a\right] = 0$$

and therefore $u_{\xi\eta} = 0$, since $b^2 \neq ac$ because λ_1 and λ_2 are distinct. We then obtain (in the usual way)

$$u = f_1(\xi) + f_2(\eta) = f_1(x + \lambda_1 y) + f_2(x + \lambda_2 y)$$

where f_1, f_2 are arbitrary $\mathbf{C}^2$ functions.

The PDE

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0$$

has the auxiliary equation

$$1 + 4\lambda + 3\lambda_2 = 0, \quad \lambda = -1, \quad \lambda = \frac{1}{3}$$

and therefore general solution

$$u(x, y) = f_1(x - y) + f_2(x - \frac{1}{3}y).$$

The conditions $u(x, o) = x, \ \ \frac{\partial u}{\partial y}(x, 0) = x$ demand that f_1, f_2 satisfy

$$f_1(x) + f_2(x) = x, \quad -f_1{}'(x) - \frac{1}{3}f_2{}'(x) = x,$$

from which we obtain

$$2f_1(x) = -\frac{3x^2}{2} - x + C, \quad f_1(x) = -\frac{3x^2}{4} - \frac{x}{2} + \frac{C}{2}, \quad f_2(x) = \frac{3x}{2} + \frac{3x^2}{4} - \frac{C}{2}$$

where C is an arbitrary constant. A straightforward calculation then gives

$$u(x, y) = -\frac{3}{4}(x - y)^2 - \frac{1}{2}(x - y) + \frac{C}{2} + \frac{3}{2}(x - \frac{y}{3}) + \frac{3}{4}(x - \frac{1}{3}y)^2 - \frac{C}{2} = x + xy - \frac{2y^2}{3}.$$

2. Given the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

consider the trial solution $\widetilde{u} = X(x)T(t)$. Separation of variables immediately gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{d^2T}{dt^2} = -\omega^2$$

so that

$$\widetilde{u} = (A_\omega \cos \omega x + B_\omega \sin \omega x)(C_\omega \cos \omega t + D_\omega \sin \omega t) \tag{17.5}$$

A standard argument then shows that the conditions $u(0, t) = u(2, t) = 0 \ \forall t \geq 0$ demand that $A_\omega = 0, \ \ 2\omega = n\pi$, where n is an integer. We have generated a class of solutions given by $\widetilde{u}_n = \sin \frac{n\pi x}{2}[E_n \cos \frac{n\pi t}{2} + F_n \sin \frac{n\pi t}{2}]$. The condition $\frac{\partial u}{\partial t}(x, 0) = 0$ will be satisfied by taking $F_n = 0$, so three of the four conditions are satisfied by functions in the class given by

$$\widetilde{u}_n = E_n \sin \frac{n\pi x}{2} \cos \frac{n\pi t}{2}$$

The final condition $u(x,0) = \sin(\pi x), \quad 0 \le x \le 2$ demands that $n = 2, \; E_2 = 1$ so the required solution is $u(x,t) = \sin(\pi x)\cos(\pi t)$.

In the final part of the question we have to solve the wave equation subject to the conditions $u(x,0) = hx, \quad 0 \le x \le 1, \quad u(x,0) = \frac{h}{3}(4-x), \quad 1 \le x \le 4, \quad u(0,t) = u(4,t) = 0 \; \forall t \ge 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le 4.$

Starting with solutions of the type (17.5) the condition $\frac{\partial u}{\partial t} = 0$ demands $D_\omega = 0$, the condition $u(0,t) = 0$ requires $A_\omega = 0$. This leaves us with a class of solutions of the type $\tilde{u} = E_\omega \sin \omega x \cos \omega t$. The condition $u(4,t) = 0 \; \forall t \ge 0$ requires us to choose ω so that $4\omega = n\pi$, where n is an integer. We see therefore, that 3 of the 4 conditions are satisfied by solutions of the type $\tilde{u}_n(x,t) = E_n \sin(\frac{n\pi x}{4})\cos(\frac{n\pi t}{4}), \quad n = 1,2,3,\ldots$ Bearing in mind the superposition principle we try to satisfy the final condition by

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin(\frac{n\pi x}{4})\cos(\frac{n\pi t}{4})$$

so that

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin(\frac{n\pi x}{4}), \quad 0 \le x \le 4.$$

The standard Fourier method now gives

$$E_n \int_0^4 \sin^2(\frac{n\pi x}{4})\, dx = \int_0^4 u(x,0)\sin(\frac{n\pi x}{4})\, dx$$

from which we obtain

$$\frac{6E_n}{h} = 3\int_0^1 x \sin(\frac{n\pi x}{4})\, dx + \int_1^4 (4-x)\sin(\frac{n\pi x}{4})\, dx.$$

Using the formula

$$\int x \sin(\frac{n\pi x}{4})\, dx = -\frac{4x}{n\pi}\cos(\frac{n\pi x}{4}) + \left(\frac{4}{n\pi}\right)^2 \sin(\frac{n\pi x}{4})$$

(derived by integration by parts) we find that

$$E_n = \frac{32h}{3\pi^2}\frac{\sin(n\pi/4)}{n^2}, \quad u(x,t) = \frac{32h}{3\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi/4)}{n^2}\sin(\frac{n\pi x}{4})\cos(\frac{n\pi t}{4}).$$

3. This question was on Legendre polynomials which are no longer part of the course.

4. Since z^n is analytic (except at $z = 0$ in the case where n is negative) we see that $\Re z^n = r^n \cos n\theta, \quad \Im z^n = r^n \sin n\theta$ satisfy the 2-dimensional Laplace equation. We can therefore try to fit the given boundary conditions $u(r, \theta = 0) = 0, \quad u(r, \theta = \pi/6) = 0,$

$0 \leq r < a, \quad u(a,\theta) = C, \quad 0 < \theta < \pi/6$ with

$$u(r,\theta) = \sum_n (A_n r^n \cos n\theta + B_n r^n \sin n\theta).$$

The first condition is satisfied if we choose $A_n = 0$, so that $u(r,\theta) = \sum_n B_n r^n \sin n\theta$. The second condition will then be satisfied if we choose $n\pi/6 = m\pi, \; m = 1, 2, 3, \ldots$ i.e. $n = 6m$ which leaves us with $u(r,\theta) = \sum_{m=1}^{\infty} B_m r^{6m} \sin(6m\theta)$. The third condition, $u(a,\theta) = C, \; 0 < \theta < \pi/6$ requires that

$$C = \sum_{m=1}^{\infty} B_m a^{6m} \sin(6m\theta), \quad 0 < \theta < \frac{\pi}{6}$$

The standard Fourier method now gives

$$C \int_0^{\frac{\pi}{6}} \sin(6m\theta)\, d\theta = B_m a^{6m} \int_0^{\frac{\pi}{6}} \sin^2(6m\theta)\, d\theta = B_m a^{6m} \frac{\pi}{12}$$

We therefore have

$$B_m = \frac{2C}{m\pi a^{6m}}(1 - \cos m\pi)$$

so that B_m is zero for all even values of m. We can express the final result as

$$u(r,\theta) = \frac{4C}{\pi} \sum_{k=0}^{\infty} \left(\frac{r}{a}\right)^{6(2k+1)} \frac{\sin 6(2k+1)\theta}{2k+1}.$$

Now to the second part of the question. Clearly $u_n(x,y) = \sinh(n\pi x)\sin(n\pi y)$ satisfy the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for each n. Three of the boundary conditions are obviously satisfied by the functions u_n and, bearing in mind that Laplace's equation is linear homogeneous, we try to satisfy the fourth condition $u(1,y) = 1, \; 0 < y < 1$ with a solution of the form

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sinh(n\pi x)\sin(n\pi y)$$

To this end we require

$$1 = \sum_{n=1}^{\infty} E_n \sinh(n\pi)\sin(n\pi y), \quad 0 < y < 1$$

and

$$\int_0^1 \sin(n\pi y)\, dy = E_n \frac{1}{2}\sinh(n\pi), \quad E_n = \frac{2}{\sinh(n\pi)} \frac{(1 - \cos n\pi)}{n\pi}$$

from which the stated result follows.

[We observe that the E_n are zero when n is even]

Note: In the first part of the question one can apply separation of variables to the given PDE (Laplace's equation in plane polar coordinates). A solution is given by $u = R(r)\Theta(\theta)$ where

$$\frac{1}{R}R'' + \frac{1}{rR}R' + \frac{1}{r^2}\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$

which (by separation of variables) yields

$$\frac{d^2\Theta}{d\theta^2} + \omega^2\Theta = 0, \qquad \Theta = C_\omega \cos\omega\theta + D_\omega \sin\omega\theta$$

$$r^2R'' + rR' - \omega^2 R = 0.$$

This equation for R can be solved by trying $R = r^\lambda$ where $\lambda(\lambda-1)+\lambda-\omega^2 = 0$, so that $\lambda = \pm\omega$. On this basis we obtain a class of solutions of our PDE of the form

$$u = \sum_\omega (A_\omega r^\omega + B_\omega r^{-\omega})(C_\omega \cos\omega\theta + D_\omega \sin\omega\theta)$$

Rejecting negative powers of r (they'd lead to singularities at $r = 0$) we're led to consider solutions of the type

$$u = \sum_\omega r^\omega (C_\omega \cos\omega\theta + D_\omega \sin\omega\theta)$$

and the analysis now proceeds along the lines indicated above.

5. f is differentiable at $z_0 \in D \iff \lim_{z\to 0} \frac{f(z)-f(z_0)}{z-z_0}$ exists i.e. there is $L \in \mathbf{C}$ such that given $\epsilon > 0$ $\exists\delta > 0$, such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - L\right| < \epsilon$$

for all z such that $0 < |z - z_0| < \delta$.

f is analytic at $z_0 \in D \iff \exists r > 0$ such that f is differentiable at each point $z \in N_r(z_0)$, the open sphere, centre z_0 and radius r.

Next, using standard notation with $u = \cosh x \cos y$, $f(z) = u + iv$ the first of the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

gives

$$\frac{\partial v}{\partial y} = \sinh x \cos y, \qquad v = \sin y \sinh x + g(x)$$

whilst the second requires that g satisfies

$$\sin y \cosh x = \sin y \cosh x + g'(x), \qquad g(x) = C, \quad C \in \mathbf{R}\,.$$

The analytic function f is therefore given by

$$f(z) = \cosh x \cos y + i \sin y \sinh x + iC = \cosh x \cosh iy + \sinh x \sinh iy + iC$$
$$= \cosh(x+iy) + iC = \cosh z + iC, \quad C \in \mathbf{R}.$$

Next, let γ_1 denote the straight line from $z=0$ to $z=1$, γ_2 the straight line from $z=1$ to $z=1+i$, and γ_3 denote the straight line from $z=0$ to $z=1+i$. Then

$$\int_{\gamma_3} \overline{z}\Re z\, dz = \int_0^1 t(1-i)t(1+i)\, dt = \frac{2}{3},$$

$$\int_{\gamma_1} \overline{z}\Re z\, dz = \int_0^1 xx\, dx = \frac{1}{3},$$

$$\int_{\gamma_2} \overline{z}\Re z\, dz = \int_0^1 (1-iy)(1)idy = i\Big[y - \frac{iy^2}{2}\Big]_0^1 = \frac{1}{2} + i.$$

We observe that

$$\int_{\gamma_1+\gamma_2} \overline{z}\Re z\, dz = \frac{1}{3} + \frac{1}{2} + i = \frac{5}{6} + i \neq \int_{\gamma_3} \overline{z}\Re z\, dz.$$

This implies that $\overline{z}\Re z$ does not have a primitive in $\mathbf{C}$ (otherwise the two integrals would have to be equal)

Referring to the final part of the question we observe that in the domain $\mathcal{D}$ (see diagram) that $z\,\mathrm{Log}\,z - z$ is a primitive for $\mathrm{Log}\,z$ because $\mathrm{Log}\,z$ is continuous on $\mathcal{D}$ and $\frac{d}{dz}[z\,\mathrm{Log}\,z - z] = \mathrm{Log}\,z$. It follows that

$$\int_\gamma \mathrm{Log}\,z\,dz = \Big[z\,\mathrm{Log}\,z - z\Big]_1^i = (i\,\mathrm{Log}\,i) - i - \mathrm{Log}\,1 + 1 = i(i\frac{\pi}{2}) - i - 0 + 1 = (1 - \frac{\pi}{2}) - i.$$

6. Note that the integral is always well-defined since the points $z = 1,\ z = 2$ are excluded from the path of integration by the conditions $r \neq \sqrt{2},\ r \neq \sqrt{5}$.

For $0 < r < \sqrt{2}$

$$\int_{\gamma_r} \frac{e^{-z^2}\,dz}{(z-1)(z-2)} = 0$$

by Cauchy's theorem. For $\sqrt{2} < r < \sqrt{5}$ $z = 1$ is inside γ_r and

$$\int_{\gamma_r} \frac{e^{-z^2}\,dz}{(z-1)(z-2)} = 2\pi i \frac{e^{-1}}{(-1)} = -\frac{2\pi i}{e}.$$

For $r > \sqrt{5}$ $z = 1$ and $z = 2$ are both inside γ_r so an application of Cauchy's residue gives

$$\int_{\gamma_r} \frac{e^{-z^2}\,dz}{(z-1)(z-2)} = -\frac{2\pi i}{e} + 2\pi i \frac{e^{-4}}{1} = 2\pi i\Big(\frac{1}{e^4} - \frac{1}{e}\Big).$$

Next, let $f(z) = \frac{e^{iz}}{(1+z^2)^2}$. Then f has poles at $z = \pm i$ but only $z = i$ is inside the contour C_R. We can write

$$f(z) = \frac{g(z)}{(z-i)^2}, \qquad g(z) = \frac{e^{iz}}{(z+i)^2}.$$

Then g is analytic at $z = i$ and near $z = i$

$$g(z) = g(i) + (z-i)g^{(1)}(i) + \cdots$$

so the Laurent expansion of f near $z = i$ is given by

$$f(z) = \frac{g(i)}{(z-i)^2} + \frac{g^{(1)}(i)}{(z-i)} + \cdots$$

The residue of f at $z = i$ is therefore $g^{(1)}(i)$. Now

$$g^{(1)}(i) = \frac{d}{dz}\frac{e^{iz}}{(z+i)^2}\Big|_{z=i} = -\frac{i}{2e}$$

(after a short computation)

On the assumption that $\int_{\gamma_R} \frac{e^{iz}\,dz}{(z^2+1)^2} \to 0$ as $R \to \infty$ it follows, from Cauchy's residue theorem, that

$$\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{(1+x^2)^2} = 2\pi i\Big(\frac{-i}{2e}\Big) = \frac{\pi}{e}.$$

We conclude, by a standard argument, that

$$\int_0^{\infty} \frac{\cos x\,dx}{(1+x^2)^2} = \frac{\pi}{2e}.$$

We can justify the assumption that $\int_{\gamma_R} \frac{e^{iz}\,dz}{(z^2+1)^2} \to 0$ as $R \to \infty$ as follows: We observe that

$$\Big|\int_{\gamma_R}\Big| = \Big|\int_0^{\pi} \frac{e^{iR[\cos\theta + i\sin\theta]}}{(R^2e^{2i\theta}+1)^2} iRe^{i\theta}\,d\theta\Big| \le R\int_0^{\pi} \frac{e^{-R\sin\theta}\,d\theta}{|(R^2e^{2i\theta}+1)^2|} \le R\int_0^{\pi} \frac{d\theta}{|R^2e^{2i\theta}+1|^2}.$$

Using $||z_1|-|z_2|| \le |z_1 - z_2|$ we see that

$$\Big|\int_{\gamma_R}\Big| \le R\int_0^{\pi} \frac{d\theta}{||R^2e^{2i\theta}|-1|^2} = \frac{\pi R}{(R^2-1)^2} \to 0$$

as $R \to \infty$.

The stated result has been established.

17.5 CM211A Examination Questions — June 2000

1. Show that the map $(x,y) \mapsto (\xi,\eta)$, where $\xi = x+y,\ \ \eta = x+2y$ transforms the partial differential equation (PDE)

$$2\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 2(x+y)e^{-(x+y)^2} \tag{17.6}$$

to

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = -2\xi e^{-\xi^2}.$$

Deduce that the general solution of the PDE (17.6) is

$$u(x,y) = (x+2y)e^{-(x+y)^2} + f(x+y) + g(x+2y),$$

where f, g are arbitrary $\mathbf{C}^2$ functions.

Hence show that the solution of the PDE (17.6) subject to the conditions

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0, \qquad x \in \mathbf{R},$$

is

$$u(x,y) = -xe^{-(x+y)^2} + (x+2y)e^{-(x+2y)^2} + 2\int_{x+y}^{x+2y} t^2 e^{-t^2}\,dt - 2\int_{x+y}^{x+2y} e^{-t^2}\,dt.$$

2. This question involved Legendre polynomials which are no longer part of the course.

3. (a) Show that θ, $r^n \cos n\theta$, $r^n \sin n\theta$ $(n = 1, 2, 3, \ldots)$ satisfy the two-dimensional Laplace equation (expressed in polar coordinates (r,θ))

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{17.7}$$

Hence solve equation (17.7) for $u(r,\theta)$ in the space $0 \le r < a$, $0 < \theta < \pi$ subject to the conditions

$$u(r,0) = 0, \quad u(r,\pi) = 0, \quad 0 \le r < a, \qquad u(a,\theta) = \theta, \quad 0 < \theta < \pi.$$

(b) Show that the solution $u(x,t)$ of the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

subject to the conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(a,t) = 0, \quad t > 0, \qquad u(x,0) = x, \quad 0 \le x \le a$$

is

$$u(x,t) = \frac{a}{2} - \frac{4a}{\pi^2}\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2}\cos\Big(\frac{(2r+1)\pi x}{a}\Big)e^{-\dfrac{(2r+1)^2\pi^2 t}{a^2}} \tag{17.8}$$

Prove rigorously, starting from equation (17.8), that

$$\lim_{t\to\infty} u(x,t) = \frac{a}{2}, \quad 0 \le x \le a.$$

4. (i) Show that the solution $u(x,t)$ of the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le 1, \quad t \ge 0$$

subject to the conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \ge 0,$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad u(x,0) = x(1-x), \quad 0 \le x \le 1$$

is

$$u(x,t) = \frac{8}{\pi^3}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^3}\sin[(2r+1)\pi x]\cos[(2r+1)\pi t].$$

Verify, by making an appropriate choice of x and t, that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}.$$

(ii) Solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad x \in \mathbf{R}, \quad t \geq 0$$

subject to the conditions

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad u(x,0) = \phi(x), \quad x \in \mathbf{R}\,.$$

5. (i) Let $f : D \to \mathbf{C}$, where D is a domain. Explain what is meant by the statements:

(a) f is *differentiable* at $z_0 \in D$

(b) f is *analytic* at $z_0 \in D$.

(ii) Starting from the definition of differentiability of a function of a complex variable show that

$$f : \mathbf{C} \to \mathbf{C}, \qquad f(z) = |z|^2$$

is differentiable at $z = 0$ and that $f'(0) = 0$. Does $f''(0)$ exist? State your reasons.

(iii) Use any suitable property of analytic functions to show that $u(x,y) = x^2y^2$ cannot be the real part of an analytic function of a complex variable $z = x + iy$.

(iv) Find, in terms of $z = x + iy$, the most general analytic function whose real part is $e^x \sin y$.

(v) Evaluate the integrals

$$\int z^2\,dz, \quad \text{and} \quad \int |z|^2\,dz$$

taken in the positive (i.e. anti-clockwise) sense round the triangle whose vertices are the points $z = 0,\ z = 1,\ z = i$.

6. (i) Use Cauchy's residue theorem to evaluate

$$\int_{\gamma_1}\left(z + \frac{1}{z}\right)^{2n}\frac{dz}{z},$$

where n is a positive integer and γ_1 is the unit circle, centre $z = 0$ and radius 1, parametrised by $\gamma_1(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$. Deduce that

$$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}.$$

Hint: Note that

$$(a+b)^{2n} = \sum_{r=0}^{2n} \frac{(2n)!}{(2n-r)!r!} a^r b^{2n-r}$$

for any $a, b \in \mathbf{C}$.

(ii) Find the poles of the function $f(z) = ze^{iz}/\cosh \pi z$ and show that the residue of f at $z = i/2$ is $1/(2\pi e^{1/2})$.

By evaluating

$$\int \frac{ze^{iz}\, dz}{\cosh \pi z}$$

taken round the contour shown in the diagram, prove that

$$\int_{-\infty}^{\infty} \frac{x \sin x\, dx}{\cosh \pi x} + \frac{1}{e+1}\int_{-\infty}^{\infty} \frac{\cos x\, dx}{\cosh \pi x} = \frac{1}{2}\operatorname{sech}\left(\frac{1}{2}\right).$$

(You may assume *without proof* that $\int \frac{ze^{iz}\, dz}{\cosh \pi z}$, taken along either of the sides parallel to the imaginary axis, tends to zero as $R \to \infty$.)

17.6 Solutions

1. We have $\xi = x + y$, $\eta = x + 2y$. The chain rule gives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + 2\frac{\partial u}{\partial \eta}.$$

The given PDE now becomes

$$2\Big(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\Big)\Big(\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}\Big)-3\Big(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\Big)\Big(\frac{\partial u}{\partial \xi}+2\frac{\partial u}{\partial \eta}\Big)+\Big(\frac{\partial}{\partial \xi}+2\frac{\partial}{\partial \eta}\Big)\Big(\frac{\partial u}{\partial \xi}+2\frac{\partial u}{\partial \eta}\Big) = 2\xi e^{-\xi^2}$$

which reduces to

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} = -2\xi e^{-\xi^2}.$$

Integration of this equation with respect to ξ followed by an integration with respect to η gives

$$\frac{\partial u}{\partial \eta} = e^{-\xi^2} + g_1(\eta), \qquad u = \eta e^{-\xi^2} + g_2(\eta) + g_3(\xi), \qquad g_2(\eta) = \int g_1(\eta)\, d\eta$$

where g_1, g_2 are arbitrary $\mathbf{C}^2$ functions. In terms of the variables (x, y) this reads

$$u(x,y) = (x+2y)e^{-(x+y)^2} + g_2(x+2y) + g_3(x+y). \tag{17.9}$$

It follows that

$$\frac{\partial u}{\partial y} = -2(x+2y)(x+y)e^{-(x+y)^2} + 2e^{-(x+y)^2} + 2g_2'(x+2y) + g_3'(x+y)$$

Imposing the conditions $u(x,0) = 0$, $\frac{\partial u}{\partial y}(x,0) = 0$ now gives

$$0 = xe^{-x^2} + g_2(x) + g_3(x), \qquad 0 = -2x^2e^{-x^2} + 2e^{-x^2} + 2g_2'(x) + g_3'(x).$$

The second of these equations yields

$$C = -2\int_{x_0}^{x} t^2e^{-t^2}\, dt + 2\int_{x_0}^{x} e^{-t^2}\, dt + 2g_2(x) + g_3(x)$$

where $C = 2g_2(x_0)+g_3(x_0)$, x_0 being an arbitrary real number. We now have two equations for g_2, g_3 which we can easily solve to obtain

$$g_2(x) = C + xe^{-x^2} + 2\int_{x_0}^{x} t^2e^{-t^2}\, dt - 2\int_{x_0}^{x} e^{-t^2}\, dt,$$

$$g_3(x) = -C - 2xe^{-x^2} - -2\int_{x_0}^{x} t^2e^{-t^2}\, dt + 2\int_{x_0}^{x} e^{-t^2}\, dt.$$

Substituting these expressions into equation (17.9) gives (after a little simplification)

$$u(x,y) = -xe^{-(x+y)^2} + (x+2y)e^{-(x+2y)^2} + 2\int_{x+y}^{x+2y} t^2e^{-t^2}\, dt - 2\int_{x+y}^{x+2y} e^{-t^2}\, dt.$$

2. This question related to Legendre polynomials, a topic which is no longer covered in this course.

3. Obviously $r^n \cos n\theta$, $\;r^n \sin n\theta$ satisfy the Laplace equation (expressed in polar coordinates (r, θ) i.e.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Alternatively, note that these functions are the real and imaginary parts of z^n $\;(z = re^{i\theta})$ and therefore satisfy Laplaces's equation (except at $z = 0$ in the case where n is a negative integer.)

For the given problem let's try to satisfy the boundary conditions with

$$u(r, \theta) = A\theta + \sum_n A_n r^n \cos n\theta + \sum_n B_n r^n \sin n\theta$$

Imposing the condition $u(r, \theta = 0) = 0,\; 0 \le r < a$ demands that $A_n = 0$. Similarly , the condition $u(r, \theta = \pi) = 0,\; 0 \le r < a$ requires that $A = 0$. This leaves us with the trial solution $u(r, \theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta$. The condition $u(a, \theta) = \theta,\;\; 0 < \theta < \pi$ now gives $\theta = \sum_{n=1}^{\infty} B_n a^n \sin n\theta,\; 0 < \theta < \pi$ and the standard Fourier method now gives

$$\int_0^{\pi} \theta \sin n\theta \, d\theta = B_n a^n \int_0^{\pi} \sin^2 n\theta \, d\theta = \frac{1}{2}\pi B_n a^n.$$

Carrying out the integration by parts gives $B_n = (-1)^{n+1}\frac{2}{na^n}$ and the final result

$$u(r, \theta) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{r}{a}\right)^n \sin n\theta.$$

Next, consider the diffusion equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$. We seek solutions of the form $\widetilde{u}(x, t) = X(x)T(t)$ and use the method of separation of variables. This gives the following ordinary differential equations for X, T:

$$\frac{d^2 X}{dx^2} + \omega^2 X = 0, \qquad \frac{dT}{dt} + \omega^2 T = 0$$

so that

$$\widetilde{u}(x, t) = (A_\omega \cos \omega x + B_\omega \sin \omega x)e^{-\omega^2 t}$$

is a solution of the diffusion equation. The requirement that $\frac{\partial \widetilde{u}}{\partial x}(0, t) = 0,\;\; \forall t > 0$ implies that $B_\omega = 0$. In order to satisfy $\frac{\partial \widetilde{u}}{\partial x}(a, t) = 0\;\; \forall t > 0$ we choose $\omega a = n\pi$, where n is an integer. On this basis we are led to consider the trial solution

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right) e^{-\frac{n^2\pi^2 t}{a^2}}.$$

The condition $u(x, 0) = x,\;\; 0 \le x \le a$ requires that

$$x = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right),\quad 0 \le x \le a$$

gives,

$$\int_0^a x\,dx = A_0 a, \qquad A_n \frac{a}{2} = \int_0^a x \cos\left(\frac{n\pi x}{a}\right) dx$$

so that after a little calculation

$$A_0 = \frac{a}{2}, \qquad A_n = \frac{2a}{n^2\pi^2}[\cos n\pi - 1], \quad n \geq 1.$$

We see that the A_n are zero when n is even and when n is odd $A_{2r+1} = -\frac{4a}{(2r+1)^2\pi^2}$, $r = 0, 1, 2, \ldots$. We therefore obtain as the solution of this boundary value problem

$$u(x,t) = \frac{a}{2} - \frac{4a}{\pi^2}\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} \cos\left(\frac{(2r+1)\pi x}{a}\right) e^{-\dfrac{(2r+1)^2\pi^2 t}{a^2}}.$$

Now observe that

$$\left|\sum_{r=0}^{\infty}\right| \leq \sum_{r=0}^{\infty} e^{-\dfrac{(2r+1)^2\pi^2 t}{a^2}} \leq \sum_{r=0}^{\infty} e^{-\dfrac{(2r+1)\pi^2 t}{a^2}} = \frac{e^{-\pi^2 t/a^2}}{(1 - e^{-2\pi^2 t/a^2})}$$

after summing a geometric series. On this basis we see that our solution is such that $\lim_{t\to\infty} u(x,t) = a/2, \quad 0 \leq x \leq a$.

4 Given the PDE

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{17.10}$$

we seek solutions of the form $\widetilde{u}(x,t) = X(x)T(t)$. Substituting and applying the method of separation of variables we obtain

$$\frac{d^2X}{dx^2} + \omega^2 X = 0, \qquad \frac{d^2T}{dt^2} + \omega^2 T = 0$$

from which we derive the solution

$$\widetilde{u}(x,t) = (A_\omega \cos\omega x + B_\omega \sin\omega x)(C_\omega \cos\omega t + D_\omega \sin\omega t).$$

We can satisfy the condition $\frac{\partial \widetilde{u}}{\partial t}(x,0) = 0$ by choosing $D_\omega = 0$ which leaves us with a $\widetilde{u}$ of the form

$$\widetilde{u}(x,t) = (E_\omega \cos\omega x + F_\omega \sin\omega x)\cos\omega t.$$

The condition $\widetilde{u}(0,t) = 0 \;\forall t \geq 0$ demands that $E_\omega = 0$ whilst the condition $\widetilde{u}(1,t) = 0$ is satisfied by choosing $\omega = n\pi$, where $n = 0, \pm 1, \ldots$ By this means we generate a class of solutions $\widetilde{u}_n(x,t) = F_n \sin n\pi x \cos n\pi t$ which satisfy three of the four conditions. In order to satisfy the fourth condition, $u(x,0) = x(1-x), \quad 0 \leq x \leq 1$ we therefore try

$$u(x,t) = \sum_{n=1}^{\infty} F_n \sin n\pi x \cos n\pi t.$$

The F_n must be such that $x(1-x) = \sum_{n=1}^{\infty} F_n \sin n\pi x, \quad 0 \le x \le 1$. The standard Fourier method gives

$$\frac{1}{2}F_n = \int_0^1 x(1-x)\sin n\pi x\,dx$$

which gives (after a short calculation)

$$F_n = \frac{4}{n^3\pi^3}(1-\cos n\pi)$$

and the final solution

$$u(x,t) = \frac{8}{\pi^3}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^3}\sin[(2r+1)\pi x]\cos[(2r+1)\pi t].$$

Setting $x = \frac{1}{2},\ t = 0$ gives

$$\frac{1}{4} = \frac{8}{\pi^3}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^3}\sin[(2r+1)\pi/2]$$

which may be written as

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

Next, equation (17.10) has auxiliary equation $1-\lambda^2 = 0$ and general solution

$$u(x,t) = f(x-t) + g(x+t)$$

Now, $\frac{\partial u}{\partial t} = -f'(x-t) + g'(x+t)$. The conditions $u(x,0) = \phi(x), \quad \frac{\partial u}{\partial t}(x,0) = 0$ demands that

$$\phi(x) = f(x) + g(x), \quad 0 = -f'(x) + g'(x)$$

which gives $-f(x) + g(x) = C$, where C is a parameter. We now have two equations for the functions f, g which solve to give

$$f(x) = \frac{1}{2}(\phi(x) - C), \qquad g(x) = \frac{1}{2}(\phi(x) + C)$$

which finally gives

$$u(x,t) = \frac{1}{2}[\phi(x-t) + \phi(x+t)].$$

5 (i) (a) $f : D \to \mathbf{C}$ is differentiable at $z_0 \in D$ if

$$\lim_{h\to 0}\frac{f(z_0+h) - f(z_0)}{h}$$

exists.

(b) f is analytic at $z_0 \in D$ if there exists $r > 0$ such that f is differentiable at every point $z \in N_r(z_0)$, the open disc centre z_0 and radius $r > 0$.

(ii) When $f'(z)$ exists we have

$$f'(z) = \lim_{h\to 0} \frac{|z+h|^2 - |z|^2}{h} = \lim_{h\to 0} \frac{(z+h)(\overline{z}+\overline{h}) - z\overline{z}}{h} = \lim_{h\to 0}\Big(z\frac{\overline{h}}{h} + \overline{z} + \overline{h}\Big).$$

If $z = 0$ this limit exists and is equal to zero, so $f'(0) = 0$. However, if $z \neq 0$ the limit does not exist because $\lim_{h\to 0} \overline{z}$ and $\lim_{h\to 0} \overline{h}$ both exist but $\lim_{h\to 0} \frac{\overline{h}}{h}$ doesn't exist e.g. for $h \in \mathbf{R}$, $\frac{\overline{h}}{h} = 1$; for $h = i\gamma$, $\gamma \in \mathbf{R}$, $\frac{\overline{h}}{h} = -1$. Now, every open disc centre zero contains such points h and it follows that $\lim_{h\to 0} \frac{\overline{h}}{h}$ doesn't exist. We conclude that $f''(0)$ does not exist — its existence would require $\exists f'(z),\ z \neq 0$.

(iii) We note that

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)(x^2y^2) = 2(x^2+y^2) \neq 0.$$

It follows that x^2y^2 cannot be the real part of an analytic function.

(iv) Set $f(z) = u + iv$ with $u = e^x \sin y$. The Cauchy-Riemann equations state that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which gives

$$e^x \sin y = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -e^x \cos y.$$

The first of these equations gives $v = -e^x \cos y + f_1(x)$ and substituting this in the second gives

$$-e^x \cos y + f_1'(x) = -e^x \cos y, \qquad f_1(x) = C, \quad C \in \mathbf{R}.$$

Hence

$$f(z) = e^x \sin y + i(-e^x \cos y + C) = -ie^z + iC, \quad C \in \mathbf{R}.$$

(v) $\int z^2\, dz = 0$ by Cauchy's theorem (since z^2 is analytic).

Also,

$$\int_{\gamma_1} |z|^2\, dz = \int_0^1 x^2\, dx = \frac{1}{3}$$

where γ_1 is the straight line from $z = 0$ to $z = 1$. γ_2, the straight line from $z = 1$ to $z = i$ can be parametrised by $\gamma_2(t) = (1-t) + it,\quad 0 \leq t \leq 1$. so that

$$\int_{\gamma_2} |z|^2\, dz = \int_0^1 [(1-t)^2 + t^2](i-1)\, dt = \frac{2}{3}(i-1).$$

Next, writing γ_3 for the straight line from $z = i$ to $z = 0$ we have

$$\int_{\gamma_3} |z|^2 \, dz = \int_1^0 y^2(i) \, dy = -\frac{i}{3}.$$

Hence

$$\int |z|^2 \, dz = \frac{1}{3} + \frac{2}{3}(i-1) - \frac{i}{3} = \frac{1}{3}(i-1).$$

6 (i) The residue of

$$\frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}$$

at $z = 0$ is the constant term in the binomial expansion of $(z + z^{-1})^{2n}$. Now,

$$\left(z + \frac{1}{z}\right)^{2n} = \sum_{r=0}^{2n} \frac{(2n)!}{r!(2n-r)!} z^r \left(\frac{1}{z}\right)^{2n-r}$$

and the constant term (corresponding to $r = n$) is $\frac{(2n)!}{(n!)^2}$. By the residue theorem,

$$\int_{\gamma_1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = 2\pi i \frac{(2n)!}{(n!)^2}$$

from which we obtain

$$\int_0^{2\pi} \frac{ie^{i\theta} \, d\theta}{e^{i\theta}} (e^{i\theta} + e^{-i\theta})^{2n} \, d\theta = 2\pi i \frac{(2n)!}{(n!)^2}$$

and

$$\int_0^{2\pi} \cos^{2n} \theta \, d\theta = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}.$$

(ii) The function f given by

$$f(z) = \frac{ze^{iz}}{\cosh \pi z}$$

has simple poles where $\cosh \pi z = 0$ i.e. where $\cos(i\pi z) = 0$ so that $i\pi z = \pm\frac{\pi}{2} + 2k\pi$, $z = \pm\frac{i}{2} - 2ki$ $(k = 0, \pm1, \pm2, \ldots)$ The only one of these poles which is actually inside the integration contour is $z = \frac{i}{2}$. The residue of f at this pole is equal to

$$\frac{(i/2)e^{-1/2}}{\pi \sinh i\pi/2} = \frac{(i/2)e^{-1/2}}{i\pi \sin \pi/2} = \frac{1}{2\pi e^{1/2}}$$

Applying Cauchy's residue theorem to the given rectangular contour we obtain

$$\int_{-R}^{R} \frac{xe^{ix} \, dx}{\cosh \pi x} + \int_0^1 \frac{(R+iy)e^{i(R+iy)}(i) \, dy}{\cosh \pi(R+iy)}$$

$$+\int_R^{-R}\frac{(x+i)e^{i(x+i)}(1)\,dx}{\cosh\pi(x+i)}+\int_1^0\frac{(-R+iy)e^{i(-R+iy)}(i)\,dy}{\cosh\pi(-R+iy)}=2\pi i\Big(\frac{1}{2\pi e^{1/2}}\Big)=\frac{i}{e^{1/2}}.$$

Write

$$I_1=\int_0^1\frac{(R+iy)e^{i(R+iy)}(i)\,dy}{\cosh\pi(R+iy)},\qquad I_2=\int_1^0\frac{(-R+iy)e^{i(-R+iy)}(i)\,dy}{\cosh\pi(-R+iy)}.$$

We're given that $\lim_{R\to\infty}I_k=0$ $(k=1,2)$ — a proof is given below. It follows that

$$\int_{-\infty}^{\infty}\frac{xe^{ix}\,dx}{\cosh\pi x}-\int_{-\infty}^{\infty}\frac{(x+i)e^{-1}e^{ix}\,dx}{\cosh\pi(x+i)}=\frac{i}{e^{1/2}}.$$

Now, $\cosh\pi(x+i)=\cosh\pi x\cosh\pi i+\sinh\pi x\sinh\pi i=-\cosh\pi x$ (using $\cosh\pi i=\cos\pi=-1$, $\sinh i\pi=i\sin\pi=0$) On this basis we obtain

$$(1+e^{-1})\int_{-\infty}^{\infty}\frac{xe^{ix}\,dx}{\cosh\pi x}+ie^{-1}\int_{-\infty}^{\infty}\frac{e^{ix}\,dx}{\cosh\pi x}=\frac{i}{e^{1/2}}.$$

Using the fact that $e^{ix}=\cos x+i\sin x$ we conclude that

$$\int_{-\infty}^{\infty}\frac{x\sin x\,dx}{\cosh\pi x}+\frac{1}{e+1}\int_{-\infty}^{\infty}\frac{\cos x\,dx}{\cosh\pi x}=\frac{1}{2}\operatorname{sech}\Big(\frac{1}{2}\Big).$$

In case this isn't clear note, for example, that $(x\cos x)/(\cosh\pi x)$ is an odd, integrable function and it follows that $\int_{-\infty}^{\infty}(x\cos x)/(\cosh\pi x)\,dx=0$. The question does not require proof that I_1 and I_2 tend to zero as R tends to infinity. Nevertheless, let's prove this for I_1 — the case of I_2 is similar. We note that

$$|I_1|\le\int_0^1\frac{|R+iy||e^{i(R+iy)}|\,dy}{|\cosh\pi(R+iy)|}\le\sqrt{R^2+1}\int_0^1\frac{dy}{|\cosh\pi(R+iy)|}$$

But $|\cosh\pi(R+iy)|^2=\cosh^2\pi R\cos^2\pi y+\sinh^2\pi R\sin^2\pi y=\cosh^2\pi R-\sin^2\pi y\ge\frac{1}{2}\cosh^2\pi R$ for all large R (i.e. for all R larger than some R_0.) This is clear since $\cosh\pi R\to\infty$ as $R\to\infty$ and therefore $\frac{1}{2}\cosh^2\pi R\ge\sin^2\pi y$ for all suitable large R, for all $y\in[0,1]$ (of course, $\sin^2\pi y\le 1,\ \forall y\in[0,1]$). We therefore see that

$$|I_1|\le\sqrt{2(R^2+1)}\int_0^1\frac{dy}{\cosh\pi R}=\frac{\sqrt{2(R^2+1)}}{\cosh\pi R}.$$

The right hand side of this inequality tends to zero as $R\to\infty$ and therefore I_1 tends to zero as $R\to\infty$. I_2 can be treated in a very similar way.

Note that one could compute

$$\int_{-\infty}^{\infty}\frac{\cos x\,dx}{\cosh\pi x}$$

by integrating $e^{iz}/(\cosh\pi z)$ round the same contour. Try this as an example now.

17.7 CM211A Examination Questions — June 2001

SECTION A

1. (i) Let $f : D \to \mathbf{C}$, where D is a domain. Explain what is meant by the statements:

(a) f is *differentiable* at $z_0 \in D$

(b) f is *analytic* at $z_0 \in D$.

(ii) Starting from the definition of differentiability of a function of a complex variable show that $f : \mathbf{C} \to \mathbf{C}, \quad f(z) = z^2|z|^2$ is differentiable at $z = 0$ but at no other point.

[25 MARKS]

2. (i) Show that $u(x, y) = x^4 - x^2y^2 + y^4, \quad (x, y) \in \mathbf{R}^2$, cannot be the real part of any analytic function of a complex variable $z = x + iy$.

(ii) Determine the most general analytic function of which $e^{-y} \sin x$ is the real part.

[25 MARKS]

3. (i) Evaluate

$$\int_\gamma \frac{e^{iz}}{z}\, dz,$$

where γ is the circle centre $z = 0$ and radius equal to unity. Deduce that

$$\int_0^{2\pi} e^{-\sin\theta} \cos(\cos\theta)\, d\theta = 2\pi.$$

(ii) Let $\Omega = \{r : r > 0, \quad r \neq 1, \ r \neq \sqrt{2}\}$.

Evaluate

$$\int_{\gamma_r} \frac{dz}{z(z-1)^2},$$

where γ_r is the circle centre i and radius r, for all $r \in \Omega$.

[25 MARKS]

4.

Use the method of separation of variables to show that the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

has solutions of the form

$$\widetilde{u}(x, y) = (A_\omega \cosh \omega x + B_\omega \sinh \omega x))(C_\omega \cos \omega y + D_\omega \sin \omega y).$$

By considering the family of solutions consisting of the functions

$$\widetilde{u}_n(x,y) = F_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right), \quad n = 1,2,3,\ldots$$

or otherwise, show that the solution of Laplace's equation in the space $0 < x < a, \quad 0 < y < a$ subject to the boundary conditions

$$u(x,0) = 0 \quad 0 \leq x < a, \quad u(x,a) = 0, \quad 0 \leq x < a,$$

$$u(0,y) = 0, \quad 0 \leq y \leq a, \quad u(a,y) = y, \quad 0 < y < a$$

is

$$u(x,y) = 2a\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sinh(\frac{n\pi x}{a})}{\sinh n\pi}\frac{\sin(\frac{n\pi y}{a})}{n\pi}.$$

[25 MARKS]

SECTION B

5. (i) Show that the Fourier transform of the function f given by

$$f(x) = \frac{1}{1+x^2}, \; x \in \mathbf{R},$$

is $\widetilde{f}$, where

$$\widetilde{f}(k) = \pi e^{-k} \text{ for } \; k \geq 0.$$

Assuming that

$$\widetilde{f}(k) = \pi e^{-|k|} \text{ for } k \in \mathbf{R},$$

verify the Fourier inversion formula for f.

Hint: Suppose $k > 0$ and consider the integral $\int_\gamma \frac{e^{ikz}}{1+z^2}\,dz$, where γ is the contour which consists of the portion of the real-axis from $x = -R$ to $x = R$ together with the semi-circular arc γ_R given by $\gamma_R(\theta) = Re^{i\theta}, \; 0 \leq \theta \leq \pi$. You may assume that $\int_{\gamma_R} \frac{e^{ikz}}{1+z^2}\,dz$ tends to zero as R tends to infinity.

(ii) Consider the partial differential equation

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial u}{\partial t}, \quad x \in \mathbf{R}, \quad t > 0$$

subject to the conditions

$$u(x,0) = f(x), \quad x \in \mathbf{R}, \qquad u, \; \frac{\partial u}{\partial x}, \; \frac{\partial^2 u}{\partial x^2} \to 0, \text{ as } \; |x| \to \infty, \quad \forall t > 0.$$

Use the method of Fourier transforms to show that the solution is

$$u(x,t) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{\infty} \mathrm{Ai}\Big(\frac{z-x}{(3t)^{1/3}}\Big) f(z)\,dz$$

where the Airy function Ai is defined by

$$\mathrm{Ai}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(z\zeta + \frac{1}{3}\zeta^3)}\,d\zeta.$$

[50 MARKS]

6.

Answer part (a) or part (b) of this question, not both.

(a) Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$, where $\gamma_1(x) = x, \;\; 0 \le x \le R$, $\gamma_2(t) = R + tR(i-1)$, $0 \le t \le 1$, and $\gamma_3(y) = iy, R \ge y \ge 0$. Sketch the trace of γ.

Evaluate

$$\int_\gamma \frac{e^{iz}}{z+a}\,dz, \qquad \text{where } a > 0.$$

Prove that

$$\lim_{R\to\infty} \int_{\gamma_2} \frac{e^{iz}}{z+a}\,dz = 0.$$

[**Hint**: You may *assume* that for all $t,\; 0 \le t \le 1$,

$$(R + a - tR)^2 + t^2R^2 \ge \frac{1}{2}(R+a)^2.]$$

Hence show that

$$\int_0^\infty \frac{\cos x\,dx}{x+a} = \int_0^\infty \frac{ye^{-y}}{a^2+y^2}\,dy, \qquad \int_0^\infty \frac{\sin x\,dx}{x+a} = \int_0^\infty \frac{ae^{-y}}{a^2+y^2}\,dy$$

and deduce that

$$0 < \int_0^\infty \frac{\cos x\,dx}{x+a} \le \frac{1}{a^2}, \qquad 0 < \int_0^\infty \frac{\sin x\,dx}{x+a} \le \frac{1}{a}.$$

[50 MARKS]

(b) By considering

$$\int_{C_R} \frac{\mathrm{Log}(1-iz)}{z^2 - 2z\sin\alpha + 1}\,dz, \quad 0 \le \alpha < \pi/2,$$

where Log is the principal value of the logarithm, and C_R is the contour which consists of the portion of the real-axis from $x = -R$ to $x = R$ together with the semi-circular arc γ_R given by $\gamma_R(\theta) = Re^{i\theta}, \quad 0 \le \theta \le \pi$, show that for $0 \le \alpha < \pi/2$

$$\int_{-\infty}^{\infty} \frac{\arctan x \, dx}{(x^2 - 2x\sin\alpha + 1)} = \frac{\pi\alpha}{2\cos\alpha}, \quad \int_{-\infty}^{\infty} \frac{\ln(1+x^2)\, dx}{(x^2 - 2x\sin\alpha + 1)} = \frac{2\pi\ln(2\cos(\alpha/2))}{\cos\alpha}.$$

[50 MARKS]

7.

(i) Given that the Legendre polynomial $P_n(x)$ of degree n satisfies the differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

prove that the Legendre polynomials are orthogonal over the interval $[-1, 1]$ in the sense that

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0, \qquad m \ne n.$$

(ii) Suppose that u is a $\mathbf{C}^2$ function which satisfies the 3-dimensional Laplace equation in the space D defined by $a < r < b$, where (r, θ, ψ) are spherical polar coordinates. Suppose that the boundary conditions are

$$u(a, \theta, \psi) = 0, \quad u(b, \theta, \psi) = \cos^3\theta + \cos^4\theta, \ 0 \le \theta \le \pi, \ 0 \le \psi \le 2\pi.$$

Find $u(r, \theta, \psi)$ in D. You may assume the following information:

- Laplace's equation in 3-dimensions has solutions of the form $r^n P_n(\cos\theta)$, $r^{-(n+1)}P_n(\cos\theta), \quad n = 0, 1, 2, 3, \ldots$
- The Legendre polynomials P_0, P_1, P_2, P_3, P_4 are given by the formulae

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

[50 Marks]

17.8 Solutions

Question 1 (i) f is differentiable at $z_0 \in D \iff$

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists.}$$

f is analytic at $z_0 \in D \iff \exists r > 0$ such that f is differentiable at every point z in the open disc $N_r(z_0)$, centre z_0 and radius $r > 0$.

(ii) For $f(z) = z^2|z|^2$ we have

$$\frac{f(z+h)-f(z)}{h} = \frac{(z+h)^3(\overline{z}+\overline{h}) - z^2|z|^2}{h} = \frac{(z^3+3z^2h+3zh^2+h^3)(\overline{z}+\overline{h}) - z^2|z|^2}{h}$$

$$= \frac{3z^2\overline{z}h + 3z\overline{z}h^2 + h^3\overline{z} + z^3\overline{h} + 3z^2h\overline{h} + 3zh^2\overline{h} + h^3\overline{h}}{h}$$

$$= 3z^2\overline{z} + 3z\overline{z}h + h^2\overline{z} + z^3\frac{\overline{h}}{h} + 3z^2\overline{h} + 3z|h|^2 + h|h|^2.$$

All the terms in this expression have a limit as $h \to 0$ except possibly $z^3\frac{\overline{h}}{h}$. We see that if $z = 0$

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = 0, \quad f'(0) = 0,$$

If $z \neq 0$

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

does not exist since $z^3\frac{\overline{h}}{h}$ does not have a limit as $h \to 0$ (e.g. for $h = \gamma,\ \gamma \in \mathbf{R},\ \frac{\overline{h}}{h} = 1$, whereas for $h = i\gamma,\ \gamma \in \mathbf{R},\ \frac{\overline{h}}{h} = -1$)

Question 2 (i) Note that

$$\frac{\partial}{\partial x}u(x,y) = 4x^3 - 2xy^2, \quad \frac{\partial^2}{\partial x^2}u(x,y) = 12x^2 - 2y^2,$$

$$\frac{\partial^2}{\partial y^2}u(x,y) = 12y^2 - 2x^2, \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x,y) = 10(x^2+y^2) \neq 0.$$

Since u does not satisfy Laplace's equation it cannot be the real part of any analytic function of $z = x + iy$.

(ii) Suppose the analytic function is $f(z) = u(x,y) + iv(x,y)$. The Cauchy-Riemann equations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

With $u = e^{-y}\sin x$ these give

$$\frac{\partial v}{\partial y} = e^{-y}\cos x, \quad \frac{\partial v}{\partial x} = e^{-y}\sin x.$$

The first of these equations gives $v = -e^{-y}\cos x + g(x)$ and substituting this expression for v in the second gives

$$e^{-y}\sin x + g'(x) = e^{-y}\sin x, \qquad g(x) = C, \quad C \in \mathbf{R}\,.$$

We therefore obtain

$$\begin{aligned} f(z) &= -ie^{-y}(\cos x + i\sin x) + iC = -ie^{-y}e^{ix} + iC \\ &= -ie^{i(x+iy)} + iC = -ie^{iz} + iC, \quad C \in \mathbf{R} \end{aligned}$$

as the most general analytic function of which u is the real part.

Question 3 (i) $\frac{e^{iz}}{z}$ has a simple pole at $z = 0$ with residue equal to 1. It follows that

$$\int_\gamma \frac{e^{iz}}{z} = 2\pi i.$$

With $\gamma(\theta) = e^{i\theta},\ 0 \le \theta \le 2\pi$ we derive

$$\int_0^{2\pi} \frac{e^{i(\cos\theta + i\sin\theta)}}{e^{i\theta}}(ie^{i\theta})\,d\theta = 2\pi i, \quad \int_0^{2\pi} e^{i(\cos\theta + i\sin\theta)}\,d\theta = 2\pi.$$

Extracting the real part of both sides gives

$$\int_0^{2\pi} e^{-\sin\theta}\cos(\cos\theta)\,d\theta = 2\pi.$$

(ii) For $r < 1$ we have

$$\int_{\gamma_r} \frac{dz}{z(z-1)^2} = 0, \qquad \text{by Cauchy's theorem.}$$

For $1 < r < \sqrt{2}$ we have

$$\int_{\gamma_r} \frac{dz}{z(z-1)^2} = (2\pi i)\,\mathrm{Res}\frac{1}{z(z-1)^2}\bigg|_{z=0} = 2\pi i$$

whilst for $r > \sqrt{2}$

$$\int_{\gamma_r} \frac{dz}{z(z-1)^2} = 2\pi i + (2\pi i)\,\mathrm{Res}\frac{1}{z(z-1)^2}\bigg|_{z=1}.$$

Writing $\frac{1}{z(z-1)^2} = \frac{g(z)}{(z-1)^2} = \frac{g(1)+g^{(1)}(1)(z-1)+\cdots}{(z-1)^2}$ with $g(z) = \frac{1}{z}$ we see that

$$\int_{\gamma_r} \frac{dz}{z(z-1)^2} = 2\pi i + 2\pi i g^{(1)}(1) = 2\pi i - 2\pi i = 0.$$

Question 4 Substituting the trial solution $\widetilde{u}(x,y) = X(x)Y(y)$ into the PDE and dividing through by XY gives

$$\frac{1}{X}X'' + \frac{1}{Y}Y'' = 0$$

and it follows that there are solutions such that

$$X'' - \omega^2 X = 0, \quad Y'' + \omega^2 Y = 0, \quad \omega \in \mathbf{R}$$

We see that there exists a class of solutions of the form

$$\widetilde{u}(x,y) = (A_\omega \cosh \omega x + B_\omega \sinh \omega x)(C_\omega \cos \omega y + D_\omega \sin \omega y).$$

The solutions

$$\widetilde{u}_n(x,y) = F_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

evidently satisfy three of the four boundary conditions, namely $u(x,0) = 0,\ 0 \le x < a,\ u(0,y) = 0,\ 0 \le y \le a,\ u(x,a) = 0, 0 \le x < a$. In order to satisfy the condition $u(a,y) = y,\ 0 < y < a$, we use the superposition principle and try

$$u(x,y) = \sum_{n=1}^{\infty} F_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right).$$

We require

$$y = \sum_{n=1}^{\infty} F_n \sinh(n\pi)\sin\left(\frac{n\pi y}{a}\right), \quad 0 < y < a$$

and the Fourier method gives

$$\int_0^a y \sin\left(\frac{n\pi y}{a}\right) dy = F_n \sinh(n\pi)\int_0^a \sin^2\left(\frac{n\pi y}{a}\right) dy = \frac{a}{2}F_n \sinh(n\pi)$$

so that

$$F_n = \frac{2}{a\sinh(n\pi)}\int_0^a y\sin\left(\frac{n\pi y}{a}\right) dy = \frac{2a(-1)^{n+1}}{(n\pi)\sinh(n\pi)}$$

and

$$u(x,y) = 2a\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sinh\left(\frac{n\pi x}{a}\right)}{\sinh(n\pi)}\frac{\sin\left(\frac{n\pi y}{a}\right)}{(n\pi)}.$$

Question 5 (i) With $f(x) = \frac{1}{1+x^2}$ we consider $\int_\gamma e^{ikz}\, dz/(1+z^2),\ \ k \ge 0$, taken along the contour γ which consists of the portion of the real axis from $x = -R$ to $x = R$ together with the semi-circular arc parametrised by $\gamma_R(\theta) = Re^{i\theta},\ 0 \le \theta \le \pi$. The integrand has simple poles at $z = \pm i$ and Cauchy's residue theorem gives

$$\int_{-R}^{R} \frac{e^{ikx}\,dx}{1+x^2} + \int_0^\pi \frac{e^{ik(R\cos\theta + iR\sin\theta)}}{1+R^2e^{2i\theta}}(iRe^{i\theta})\,d\theta = 2\pi i \frac{e^{-k}}{2i} = \pi e^{-k}.$$

It is given that the second integral tends to zero as $R \to \infty$. [This is easily proved since for $k \ge 0$ we have

$$\left|\int_0^\pi\right| \le \int_0^\pi \frac{e^{-kR\sin\theta}R\,d\theta}{|1+R^2e^{2i\theta}|} \le \int_0^\pi \frac{R\,d\theta}{|R^2e^{2i\theta}+1|} \le \int_0^\pi \frac{R\,d\theta}{||R^2e^{2i\theta}|-1|} = \frac{\pi R}{R^2-1}$$

from which the given result clearly follows in the limit $R \to \infty$.]

We therefore obtain

$$\int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{1+x^2} = \widetilde{f}(k) = \pi e^{-k}, \quad k \ge 0.$$

It is *given* (and a little reflection makes this clear) that for all real k we can write

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} \frac{e^{ikx}\,dx}{1+x^2} = \pi e^{-|k|}.$$

To check the inversion theorem we compute

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\widetilde{f}(k)\,dk &= \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ikx}\pi e^{k}\,dk + \frac{1}{2\pi}\int_0^\infty e^{-ikx}\pi e^{-k}\,dk \\ &= \frac{1}{2}\Big[\int_{-\infty}^{0} e^{k(1-ix)}\,dk + \int_0^\infty e^{-k(1+ix)}\,dk\Big] \\ &= \frac{1}{2}\Big[\frac{1}{1-ix} + \frac{1}{1+ix}\Big] = \frac{1}{1+x^2} = f(x) \end{aligned}$$

so the Fourier inversion formula is valid for f.

(ii) Next, given the PDE $\frac{\partial^3 u}{\partial x^3} = \frac{\partial u}{\partial t}$ multiply by e^{ikx} and integrate over x:

$$\int_{-\infty}^{\infty} \frac{\partial^3 u}{\partial x^3} e^{ikx}\,dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{ikx}u(x,t)\,dx = \frac{\partial}{\partial t}\widetilde{u}, \quad \widetilde{u}(k,t) = \int_{-\infty}^{\infty} e^{ikx}u(x,t)\,dx$$

Integration by parts and imposing the boundary conditions at infinity gives

$$e^{ikx}\frac{\partial^2 u}{\partial x^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(ik)e^{ikx}\frac{\partial^2 u}{\partial x^2}\,dx = \ldots = (-ik)^3\widetilde{u} = \frac{\partial}{\partial t}\widetilde{u}$$

whence

$$\widetilde{u} = Ae^{ik^3t}, \qquad A = A(k)$$

Since $u(x,0) = f(x), x \in \mathbf{R}$, it follows that $\widetilde{u}(k,0) = \int_{-\infty}^{\infty} e^{ikx} f(x)\,dx = \widetilde{f}(k)$. We therefore obtain $A(k) = \widetilde{f}(k)$ and $\widetilde{u} = \widetilde{f}(k)e^{ik^3t}$. Applying the Fourier inversion formula we derive

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\widetilde{f}(k)e^{ik^3t}\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik^3t-ikx}\,dk\Big(\int_{-\infty}^{\infty} e^{ikz}f(z)\,dz\Big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)\Big(\int_{-\infty}^{\infty} e^{ik(z-x)+ik^3t}\,dk\Big)\,dz$$

Changing the variable in the inner integral from k to ζ, where $k^3t = \frac{1}{3}\zeta^3$, so that $k = \zeta/(3t)^{1/3}$, we obtain

$$\int_{-\infty}^{\infty} e^{ik(z-x)+ik^3t}\,dk = \frac{1}{(3t)^{1/3}}\int_{-\infty}^{\infty} e^{i\frac{\zeta(z-x)}{(3t)^{1/3}}+i\frac{1}{3}\zeta^3}\,d\zeta = \frac{2\pi}{(3t)^{1/3}}\mathrm{Ai}\Big(\frac{z-x}{(3t)^{1/3}}\Big),$$

where Ai denotes the Airy function defined by

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(z\zeta+\frac{1}{3}\zeta^3)}\,d\zeta.$$

The final result is then

$$u(x,t) = \frac{1}{(3t)^{1/3}}\int_{-\infty}^{\infty} f(z)\mathrm{Ai}\Big(\frac{z-x}{(3t)^{1/3}}\Big)\,dz.$$

Question 6 (a) $e^{iz}/(z+a)$ has a simple pole at $z=-a$ but this point is not inside the contour of integration so by Cauchy's theorem

$$\int_\gamma \frac{e^{iz}\,dz}{z+a} = 0.$$

It follows that

$$\int_0^R \frac{e^{ix}\,dx}{x+a} + \int_0^1 \frac{e^{i(R+tR(i-1))}R(i-1)\,dt}{R+a+tR(i-1)} + \int_R^0 \frac{e^{-y}(i\,dy)}{iy+a} = 0,$$

via the parametrisation $\gamma_2(t) = R + tR(i-1), \quad 0 \le t \le 1$. Now,

$$\Big|\int_0^1 \frac{e^{i(R+tR(i-1))}R(i-1)\,dt}{R+a+tR(i-1)}\Big| \le \int_0^1 \frac{e^{-tR}R\sqrt{2}\,dt}{\sqrt{(R+a-tR)^2+t^2R^2}}.$$

It's given that $(R+a-tR)^2 + t^2R^2 \ge \frac{1}{2}(R+a)^2$, as can easily be verified:

$[(R+a-tR)^2 + t^2R^2 = (R+a)^2 + 2[(tR) - \frac{1}{2}(R+a)]^2 - \frac{1}{2}(R+a)^2$
$= \frac{1}{2}(R+a)^2 + 2[\ldots]^2 \ge \frac{1}{2}(R+a)^2]$

Consequently,

$$\left|\int_0^1\right| \le 2\int_0^1 \frac{Re^{-tR}\,dt}{R+a} = \frac{2}{R+a}(1-e^{-R}) \to 0 \text{ as } R\to\infty.$$

We conclude that

$$\int_0^\infty \frac{e^{ix}\,dx}{x+a} = \int_0^\infty \frac{ie^{-y}\,dy}{iy+a} = \int_0^\infty \frac{e^{-y}(y+ia)\,dy}{a^2+y^2}$$

and that

$$\int_0^\infty \frac{\cos x\,dx}{x+a} = \int_0^\infty \frac{ye^{-y}\,dy}{a^2+y^2}$$

and

$$\int_0^\infty \frac{\sin x\,dx}{x+a} = \int_0^\infty \frac{ae^{-y}\,dy}{a^2+y^2}.$$

Since

$$0 < \int_0^\infty \frac{ye^{-y}\,dy}{a^2+y^2} \le \frac{1}{a^2}\int_0^\infty ye^{-y}\,dy = \frac{1}{a^2}, \text{ and } 0 < \int_0^\infty \frac{ae^{-y}\,dy}{a^2+y^2} \le \frac{1}{a}\int_0^\infty e^{-y}\,dy = \frac{1}{a}$$

the stated result follows.

(b) $\mathrm{Log}(1-iz)$ has a branch point at $z=-i$ but this poses no problems as we integrate round the contour C_R. We note that $z^2-2z\sin\alpha+1=0$ when $(z-\sin\alpha)^2+(1-\sin^2\alpha)=0$ i.e $z=i\cos\alpha+\sin\alpha$ or $z=-i\cos\alpha+\sin\alpha$. For $0\le\alpha<\frac{\pi}{2}$ only $z=i\cos\alpha+\sin\alpha=ie^{-i\alpha}$ is inside the contour. We have

$$\mathrm{Res}\frac{\mathrm{Log}(1-iz)}{z^2-2z\sin\alpha+1}\Big|_{z=ie^{-i\alpha}} = \frac{\mathrm{Log}(1+e^{-i\alpha})}{2ie^{-i\alpha}-2\sin\alpha} = \frac{\mathrm{Log}(1+e^{-i\alpha})}{2i\cos\alpha}$$

$$= \frac{\mathrm{Log}\, e^{-\frac{1}{2}i\alpha}(e^{i\alpha/2}+e^{-i\alpha/2})}{2i\cos\alpha} = \frac{\ln(2\cos\alpha/2)-i\alpha/2}{2i\cos\alpha}.$$

Cauchy's residue theorem, applied to the contour C_R now gives

$$\int_{-R}^{R}\frac{\mathrm{Log}(1-ix)\,dx}{x^2-2x\sin\alpha+1} + \int_0^\pi \frac{\mathrm{Log}(1-Re^{i\theta})(iR)e^{i\theta}\,d\theta}{R^2e^{2i\theta}-2Re^{i\theta}\sin\alpha+1} = \frac{\pi}{\cos\alpha}(\ln(2\cos\alpha/2)-i\alpha/2)$$

We note that

$$\left|\int_0^\pi\right| \le \int_0^\pi \frac{|\,\mathrm{Log}(1-Re^{i\theta})|R\,d\theta}{|R^2e^{2i\theta}-2Re^{i\theta}\sin\alpha+1|}$$

Since $|R^2e^{2i\theta}-2Re^{i\theta}\sin\alpha+1| \ge |R^2-|1-2Re^{i\theta}\sin\alpha||$ and since $|1-2Re^{i\theta}\sin\alpha| \le 1+2R\sin\alpha < 1+2R$ we have $|R^2e^{2i\theta}-2Re^{i\theta}\sin\alpha+1| \ge R^2-1-2R$ (we are interested in large R). Also $|\,\mathrm{Log}(1-Re^{i\theta})| \le |\ln(|1-iRe^{i\theta}|)|+\pi \le \ln(1+R)+\pi$ and therefore

$$\left|\int_0^\pi\right| \le \frac{\pi R(\pi+\ln(1+R))}{R^2-1-2R} \to 0 \text{ as } R\to\infty.$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{(\ln\sqrt{1+x^2} - i\arctan x)\,dx}{x^2 - 2x\sin\alpha + 1} = \frac{\pi}{\cos\alpha}(\ln(2\cos\alpha/2) - i\alpha/2)$$

and that

$$\int_{-\infty}^{\infty} \frac{\ln(1+x^2)\,dx}{x^2 - 2x\sin\alpha + 1} = \frac{2\pi}{\cos\alpha}\ln(2\cos\alpha/2), \quad 0 \le \alpha < \frac{\pi}{2},$$

$$\int_{-\infty}^{\infty} \frac{\arctan x\,dx}{x^2 - 2x\sin\alpha + 1} = \frac{\pi\alpha}{2\cos\alpha}, \quad 0 \le \alpha < \frac{\pi}{2}.$$

Question 7 (i) We observe that

$$\frac{d}{dx}[(1-x^2)P_n'] + n(n+1)P_n = 0,$$

$$\frac{d}{dx}[(1-x^2)P_m'] + m(m+1)P_m = 0.$$

Multiplying the first of these equations by P_m, the second by P_n and subtracting gives:

$$P_m D[(1-x^2)P_n'] - P_n D[(1-x^2)P_m'] + [n(n+1) - m(m+1)P_nP_m = 0, \quad D = \frac{d}{dx},$$

so that

$$D[P_m(1-x^2)P_n' - P_n(1-x^2)P_m'] + (n-m)(n+m+1)P_nP_m = 0.$$

Integration over $[-1,1]$ now gives

$$(n-m)(n+m+1)\int_{-1}^{1} P_nP_m\,dx = 0$$

so that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0, \quad n \ne m.$$

[Of course, $n+m+1$ cannot be zero since m, n are non-negative integers.]

(ii) From the expressions for the Legendre polynomials which are given in the question we easily obtain:

$$x^3 = \frac{2}{5}P_3 + \frac{3}{5}P_1, \quad x^4 = \frac{8}{35}P_4 + \frac{4}{7}P_2 + \frac{1}{5}$$

so that, in terms of Legendre polynomials the boundary conditions read:

$$u(a,\theta) = 0, \quad 0 \le \theta \le \pi,$$

$$u(b,\theta) = \frac{8}{35}P_4(\cos\theta) + \frac{2}{5}P_3(\cos\theta) + \frac{4}{7}P_2(\cos\theta) + \frac{3}{5}P_1(\cos\theta) + \frac{1}{5}P_0(\cos\theta)$$

The required solution of Laplaces's equation is clearly independent of the azimuthal angle ψ so we try as the solution to our problem

$$u(r,\theta) = \sum (A_n r^n + B_n r^{-(n+1)}) P_n(\cos\theta)$$

The condition $u(a,\theta) = 0, 0 \le \theta \le \pi$ immediately gives $B_n = -a^{2n+1} A_n$ so that

$$u(r,\theta) = \sum A_n (r^n - a^{2n+1} r^{-(n+1)}) P_n(\cos\theta)$$

whilst the second condition now demands that

$$\frac{8}{35} P_4(\cos\theta) + \frac{2}{5} P_3(\cos\theta) + \frac{4}{7} P_2(\cos\theta) + \frac{3}{5} P_1(\cos\theta) + \frac{1}{5} P_0(\cos\theta)$$

$$= \sum A_n (b^n - a^{2n+1} b^{-(n+1)}) P_n(\cos\theta), \quad 0 \le \theta \le \pi$$

Hence, the solution to the problem is

$$u(r,\theta) = A_0\Big(1 - \frac{a}{r}\Big) + A_1\Big(r - \frac{a^3}{r^2}\Big) P_1(\cos\theta) + A_2\Big(r^2 - \frac{a^5}{r^3}\Big) P_2(\cos\theta)$$

$$+ A_3\Big(r^3 - \frac{a^7}{r^4}\Big) P_3(\cos\theta) + A_4\Big(r^4 - \frac{a^9}{r^5}\Big) P_4(\cos\theta),$$

where

$$A_0\Big(1 - \frac{a}{b}\Big) = \frac{1}{5}, \qquad A_0 = \frac{b}{5(b-a)},$$

$$A_1\Big(b - \frac{a^3}{b^2}\Big) = \frac{3}{5}, \qquad A_1 = \frac{3b^2}{5(b^3 - a^3)},$$

$$A_2\Big(b^2 - \frac{a^5}{b^3}\Big) = \frac{4}{7}, \qquad A_2 = \frac{4b^3}{7(b^5 - a^5)},$$

$$A_3\Big(b^3 - \frac{a^7}{b^4}\Big) = \frac{2}{5}, \qquad A_3 = \frac{2b^4}{5(b^7 - a^7)},$$

$$A_4\Big(b^4 - \frac{a^9}{b^5}\Big) = \frac{8}{35}, \qquad A_4 = \frac{8b^5}{35(b^9 - a^9)}.$$

17.9 CM211A Examination Questions — May 2002

SECTION A

1. (i) Show that the function $f : \mathbf{C} \to \mathbf{C}$, where $f(z) = \overline{z}$, is not differentiable at any point.

(ii) Show that the function $g : \mathbf{C} \to \mathbf{C}$, where $g(z) = z^2 + |z|^2$, is differentiable at $z = 0$ but at no other point. [30 MARKS]

2. (i) Determine the most general analytic function of which $\cos x \cosh y$ is the real part. Express your result in terms of $z = x + iy$.

(ii) Evaluate $\int |z|^2 \, dz$ and $\int z^2 \, dz$ where the integrals are taken *in the positive sense* round the triangle whose vertices are $z = 0, \ z = 1, \ z = i$.

(iii) A simple closed curve $\gamma : [0, 1] \to \mathbf{C}$ is such that $\gamma(0) = 0, \ \ \gamma(\frac{1}{4}) = 1$, $\gamma(\frac{1}{2}) = i, \ \ \gamma(1) = 0$. The trace of γ is the union of the sides of the triangle whose vertices are at $z = 0, \ z = 1, \ z = i$. Write down an explicit formula for $\gamma(t), \ t \in [0, 1]$.
[30 MARKS]

3. Give a brief complex variable argument to justify the assertion that an expression of the form

$$u(r, \theta) = \sum_n a_n r^n \cos n\theta + \sum_n b_n r^n \sin n\theta,$$

where (r, θ) denote plane polar coordinates, and $\sum_n$ denotes a sum over non-negative integers, is a solution of the two-dimensional Laplace equation.

Hence find the solution of the two-dimensional Laplace equation in the space $0 < r < a, \ \ 0 < \theta < \pi/4$ which satisfies the boundary conditions

$$u(r, \theta = 0) = 0, \quad u(r, \theta = \pi/4) = 0, \quad 0 \le r < a, \quad u(r = a, \theta) = 1, \quad 0 \le \theta \le \pi/4.$$

You may assume the orthogonality relations

$$\int_0^{\pi/4} \sin(4r\theta) \sin(4s\theta) \, d\theta = \frac{\pi}{8} \delta_{rs},$$

for any choice of the positive integers r, s. [30 MARKS]

4. Use the transformation $(x, y) \mapsto (\xi, \eta)$ where $\xi = x + y, \ \ \eta = x + 4y$ to show that the general solution of the partial differential equation

$$4\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \, \partial y} + \frac{\partial^2 u}{\partial y^2} = y, \qquad -\infty < x < \infty, \qquad y \ge 0$$

is

$$u(x, y) = f(x, y) + g_1(x + 4y) + g_2(x + y)$$

where $g_1, \ g_2$ are arbitrary $\mathbf{C}^2$ functions; your expression for $f(x, y)$ should be clearly stated.

Use this result to find the particular solution of the given partial differential equation which satisfies the boundary conditions

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0, \qquad -\infty < x < \infty.$$

Note: You are *not* required to reduce your particular solution to its simplest form.

[30 MARKS]

SECTION B

5. (i) Evaluate the integral

$$\int_{C_R} \frac{z^3\,dz}{(z-1)(z-4)^2}$$

where C_R is the circle centre $3i$ and radius R, for all values of R other than $R = \sqrt{10}$ and $R = 5$.

(ii) By considering the integral $\int_\gamma e^{-z^2}\,dz$, where γ denotes the rectangular contour which is bounded by $y = 0, \quad y = b \quad (b > 0), \quad x = 0, \quad x = R \quad (R > 0)$ prove that

$$\int_0^\infty e^{-x^2} \cos 2bx\,dx = \sqrt{\frac{\pi}{4}} e^{-b^2}$$

and that

$$\int_0^\infty e^{-x^2} \sin 2bx\,dx = e^{-b^2} \int_0^b e^{y^2}\,dy.$$

You may assume that

$$\int_0^\infty e^{-x^2}\,dx = \sqrt{\frac{\pi}{4}}.$$

Your argument should include a proof that the integral along an appropriate side of the rectangle tends to zero as R tends to infinity.

[50 MARKS]

6. (i) Let $f(x) = \frac{1}{(1+x^2)^2}, \quad x \in \mathbf{R}$. Compute the Fourier transform $\widetilde{f}(k)$, for $k \geq 0$, by integrating $e^{ikz}/(1+z^2)^2$ round the contour C_R in the z-plane which consists of the portion of the real axis from $-R$ to R, together with the semi-circular arc γ_R parametrised by $\gamma_R(\theta) = Re^{i\theta}, \quad 0 \leq \theta \leq \pi$.

You may assume that

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{ikz}\,dz}{(1+z^2)^2} = 0 \quad (k \geq 0).$$

(ii) Use the method of Fourier transforms to show (formally) that the solution of the partial differential equation

$$\frac{\partial u}{\partial t} + \lambda u = \frac{\partial^2 u}{\partial x^2} \quad (\lambda > 0)$$

in the space $-\infty < x < \infty, \quad t > 0$ subject to the conditions that

$$u \to 0 \text{ as } |x| \to \infty, \quad \frac{\partial u}{\partial x} \to 0 \text{ as } |x| \to \infty, \qquad u(x, t = 0) = f(x), \quad -\infty < x < \infty$$

is

$$u(x,t) = \frac{e^{-\lambda t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(z) e^{-\frac{(z-x)^2}{4t}} \, dz.$$

You may assume that

$$\int_{-\infty}^{\infty} e^{-\alpha(x+i\beta)^2} \, dx = \sqrt{\frac{\pi}{\alpha}}, \qquad \alpha > 0, \quad \beta \in \mathbf{R}\,.$$

[50 MARKS]

Chapter 18

Appendix 1

18.1 Laplacian in polar coordinates

We consider the Laplacian in cylindrical and speherical polar coordinates.

First, cylindrical polar coordinates. Suppose ϕ is a $\mathbf{C}^{(\mathbf{2})}$ function of the Cartesian variables (x, y, z). These are related to the cylindrical polar coordinates (r, θ, z) by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

and the problem is to express

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

in terms of derivatives with respect to the variables (r, θ, z). The chain rule gives

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x}.$$

The equations $r^2 = x^2 + y^2, \quad \tan\theta = y/x$ give

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \quad \sec^2\theta\frac{\partial\theta}{\partial x} = -\frac{y}{x^2}, \quad \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r},$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta, \quad \sec^2\theta\frac{\partial\theta}{\partial y} = \frac{1}{x}, \quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}.$$

The chain rule now gives

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial\phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta} \tag{18.1}$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial y} = \sin\theta\frac{\partial\phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta} \tag{18.2}$$

It follows that

$$\begin{aligned}
\frac{\partial^2\phi}{\partial x^2} &= \Big(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\Big)\Big(\cos\theta\frac{\partial\phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}\Big) \\
&= \cos^2\theta\frac{\partial^2\phi}{\partial r^2} - \sin\theta\cos\theta\Big(\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\Big) - \frac{\sin\theta}{r}\Big(\cos\theta\frac{\partial^2\phi}{\partial r\partial\theta} - \sin\theta\frac{\partial\phi}{\partial r}\Big) \\
&+ \frac{\sin\theta}{r^2}\Big(\sin\theta\frac{\partial^2\phi}{\partial\theta^2} + \cos\theta\frac{\partial\phi}{\partial\theta}\Big) \\
&= \cos^2\theta\frac{\partial^2\phi}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial\phi}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2}
\end{aligned} \tag{18.3}$$

$$\begin{aligned}
\frac{\partial^2\phi}{\partial y^2} &= \Big(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\Big)\Big(\sin\theta\frac{\partial\phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}\Big) \\
&= \sin^2\theta\frac{\partial^2\phi}{\partial r^2} + \sin\theta\cos\theta\Big(\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\Big) + \frac{\cos\theta}{r}\Big(\sin\theta\frac{\partial^2\phi}{\partial r\partial\theta} + \cos\theta\frac{\partial\phi}{\partial r}\Big) \\
&+ \frac{\cos\theta}{r^2}\Big(\cos\theta\frac{\partial^2\phi}{\partial\theta^2} - \sin\theta\frac{\partial\phi}{\partial\theta}\Big) \\
&= \sin^2\theta\frac{\partial^2\phi}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\phi}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial\phi}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2\phi}{\partial\theta^2}
\end{aligned} \tag{18.4}$$

From equations 18.3 and 18.4 and the identity $\cos^2\theta + \sin^2\theta = 1$ we obtain

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} \tag{18.5}$$

Note that this bears no resemblance to

$$\frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

which students sometimes use in exercises!

We can deduce the corresponding result for spherical polar coordinates (r, θ, ψ) using equation 18.5 In this case

$$x = r\sin\theta\cos\psi, \quad y = r\sin\theta\sin\psi, \quad z = r\cos\theta.$$

Put $R = r\sin\theta$ so that

$$x = R\cos\psi, \quad y = R\sin\psi, \quad z = z$$

with $z = r\cos\theta,\ R = r\sin\theta$. From equation 18.5

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial R^2} + \frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\phi}{\partial\psi^2} + \frac{\partial^2\phi}{\partial z^2}.$$

Bearing in mind that $z = r\cos\theta$, $R = r\sin\theta$ it follows from a further application of equation 18.5 (without the $\frac{\partial^2\phi}{\partial z^2}$ term) that

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial R} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}.$$

Finally, from equation 18.2 above, with $R = r\sin\theta$ playing the role of y in that formula, we obtain

$$\frac{\partial\phi}{\partial R} = \sin\theta\frac{\partial\phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta}$$

so that

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}.$$

Since

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = \frac{\partial^2\phi}{\partial\theta^2} + \cot\theta\frac{\partial\phi}{\partial\theta}$$

we can write

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}$$

which is the expression we've used in lectures. Note again that this bears no resemblance to

$$\frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial\psi^2}.$$

Finally, note a formula we've sometimes used:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\phi\right) = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}$$

(See the first two terms in our formula for the Laplacian)