A train air brake force model: Locomotive automatic brake valve and brake pipe flow formulations

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Proceedings of the Institution of Mechanical Engineers, Part F: Journal of Rail and Rapid Transit 2013 227: 19 originally published online 19 June 2012
DOI: 10.1177/0954409712447230

Original Article

Institution of MECHANICAL ENGINEERS

Journal of RAIL AND RAPID TRANSIT

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Proc IMechE Part F: J Rail and Rapid Transit
227(1) 19–37

DOI: 10.1177/0954409712447230
pif.sagepub.com
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Abstract

The goal of this study is to integrate an air brake model with efficient algorithms for train longitudinal force calculation that are based on trajectory coordinate formulations. The air brake model, developed in this investigation and presented in this paper and a companion paper, consists of the locomotive automatic brake valve, air brake pipe and car control unit (CCU). The proposed air brake force model accounts for the effect of the air flow in long train pipes as well as the effect of leakage and branch pipe flows. This model can be used to study the dynamic behavior of the air flow in the train pipe and its effect on the longitudinal train forces during brake application and release. The governing equations of the air pressure flow are developed using the general fluid continuity and momentum equations, simplified using the assumptions of one-dimensional isothermal flow. Using these assumptions, one obtains two coupled air velocity/pressure partial differential equations that depend on time and the longitudinal coordinate of the brake pipes. The partial differential equations are converted to a set of first-order ordinary differential equations using the finite element method. The resulting air brake ordinary differential equations are solved simultaneously with the train's second-order non-linear dynamic differential equations of motion that are based on the trajectory coordinates. The train car non-linear dynamics is defined using a body track coordinate system that follows the car motion. The body track coordinate system translation and orientation are defined in terms of one parameter that describes the distance traveled by the car. The configuration of the car with respect to its track coordinate system is described using two translation coordinates and three Euler angles. The operation modes of the brake system considered in this investigation are the brake release mode and the brake application mode that includes service and emergency brakes. A detailed model of the locomotive automatic brake valve is presented in this investigation and used to define the inputs to the air brake pipe during the simulation. A simplified model of this valve is also proposed in order to reduce the computational time of the simulation. In a companion paper, the detailed CCU formulation is presented. The coupling between the air brake, locomotive automatic brake valve, CCUs and train equations is established and used in the companion paper in the simulation of the non-linear dynamics of long trains. The air brake formulations presented in the two companion papers are implemented in a computer code called Analysis of Train/Track Interaction Forces (ATTIF) which is developed for the analysis of longitudinal train forces.

Keywords

Air brake, locomotive valves, longitudinal train dynamics, railroad vehicles, finite element

Date received: 20 June 2011; accepted: 27 March 2012

Introduction

Railroad vehicle systems are among the most commonly used methods of transportation, both for passengers and goods. Their widespread use has sparked, over the years, continuous technological developments, with the objective of achieving higher operating speeds in order to minimize cost and transportation time. Higher operating speeds, however, require a better and more sophisticated approach for the design of the railroad vehicle systems in order to avoid derailments and reduce the vibration and noise levels. Therefore, the development and use of accurate computer models for simulation of railroad vehicle systems subjected to different loading

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conditions, operating speeds, track geometries, braking and traction scenarios are necessary. Using these accurate computer models, it is possible to build virtual prototypes for the simulation and analysis of the non-linear dynamic behavior of long trains or for the simulation of detailed single or multiple car models.[1–10] Such studies will contribute significantly to a better understanding of the causes of derailments and accidents, and to a better understanding of the vibration, stability, dynamic characteristics and longitudinal shock loads of railroad vehicle systems.

Most trains operating in North America still employ pneumatic brakes.[11] For the most part, the automatic air brake system of a freight train consists of a *locomotive control unit*, a car control unit (CCU) located in each car and a *pipe* connecting all these elements as shown in Figure 1. This pipe that transfers both flowing air and brake signals will be referred to as the *brake pipe*, and the locomotive automatic brake valve will be simply called the *automatic brake valve* in this paper. The function of the automatic brake valve is to control the air pressure in the brake pipe for on-car compressed air storage as well as brake application and release for all cars. Such a control action provides the pressure control signal that propagates along the brake pipe serially reaching one car after another. In addition to the pressure in the brake pipe, the controlling locomotive's equalizing and emergency reservoir pressures are also controlled. By varying the position of the handle of the automatic brake valve, different scenarios and states of the brake system can be achieved. These states include the *brake release mode* in which the pressure in the brake pipe is increased in order to release the brakes and recharge the CCU's compressed air storage reservoirs, the *service mode* in which the pressure must be reduced in order to apply the brakes with a pressure reduction (at a service rate) that is determined by the handle position with respect to the full service position and the *emergency mode* that allows the pressure in the brake pipe to be quickly reduced by venting the air in the brake pipe to the atmosphere (The latter two, service and emergency modes, are referred to as the *brake application mode*.) Therefore, an accurate air brake model must be able to predict the response of the air flow in the brake pipe to the changes made in response to the position of the handle of the automatic brake valve. The brake valve that is modeled in this investigation is the 26C valve, which is used in trains in North America. The 26C valve also has an independent brake valve function that is not modeled in this study.

Many studies have focused on the pneumatic or air brake that is commonly used in trains in North America.[11–16] Shute et al. [17] investigated the effect of leakage on brake pipe gradients and flow rates. In all of these studies, different aspects of a train's air brake were investigated using a one-dimensional flow assumption. Gauthier[18] and Wright[19] studied the pneumatic control valve systems using a similar assumption. Wei and Lin [20] developed a computer model for the "120" control valve that is used in Chinese Railroads. In this study, the train's longitudinal dynamics due to brake application is not discussed. However, there have been studies that investigated the effect of the air brake on the train's longitudinal dynamics. Nasr and Mohammadi[21] studied the effect of the brake's delay time on the train's longitudinal forces. They employed a one-dimensional motion assumption for the train dynamics and used a simple model for the air brake. Sanborn et al.[8] used a simple brake model in which a constant propagation speed is assumed for the brake signal. Such an assumption of constant air propagation speed cannot be justified in many applications and does not allow for accurately predicting car coupler forces in severe braking scenarios.

There are also computer programs that simulate the dynamic behavior of railroad vehicle systems. For instance, TrainDy is a program developed by the International Union of Railways that can model a train's air brake system. An equivalent parameterization of the control valve is used in TrainDy instead of a complete pneumatic model of the brake system.[22] TOES is another program used in such simulations. However, it only allows for one degree of freedom for the cars. The program used in the present investigation, called ATTIF, allows for six degrees of freedom and can efficiently solve fully non-linear dynamic equations that are based on trajectory coordinate formulations.

The objective of this investigation is to integrate a dynamic air brake model with efficient non-linear longitudinal force algorithms for the train based on trajectory coordinate formulations. The proposed air brake force model used in this investigation employs continuity and momentum equations and accounts for the effect of the air flow in long train pipes as well as the effect of leakage and branch pipe flows. This model can be effectively used to study the dynamic behavior of the air flow in the train pipe and its effect on the longitudinal train forces during

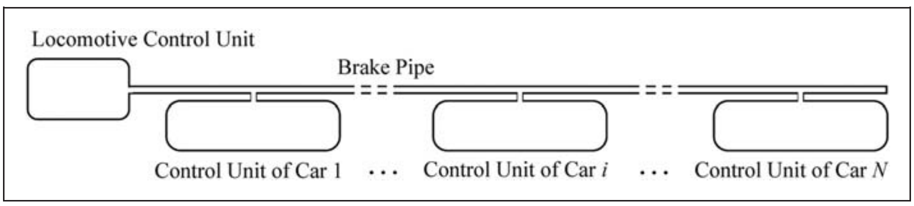

Figure 1. Main air brake components.

brake application and release. The continuity and momentum equations are simplified by using the assumptions of one-dimensional isothermal flow, leading to two coupled air velocity/pressure partial differential equations that depend on time and the longitudinal coordinate of the brake pipe. The resulting partial differential equations for the air brake are converted to a set of first-order ordinary differential equations using finite element discretization. These first-order ordinary differential equations are solved simultaneously with the train's second-order non-linear dynamic differential equations of motion that are based on the trajectory coordinates. In this investigation, the train car dynamics is defined using a body track coordinate system that follows the car motion. The translation and orientation of this coordinate system are defined in terms of one geometric trajectory parameter that describes the distance traveled by the car. The configuration of the car with respect to its track coordinate system is described using two translation coordinates and three Euler angles.[23] The non-linear trajectory coordinate formulation used in this study allows for the use of an arbitrary track geometry. The operation of the brake system, including brake application and release, is controlled by the automatic brake valve that defines the input to the air brake system during the dynamic simulation. A simplified valve model is also proposed in order to reduce the simulation's computation time. The procedure for coupling the air flow in the brake pipe, the automatic brake valve, CCU and the train equations is established and used in the simulation of the non-linear dynamics of long trains.

Air flow equations

In this section, the basic continuum mechanics equations used in this investigation to study the air flow dynamics in a train's brake pipes are presented. These equations include the momentum, continuity and constitutive equations. In the following section, the general three-dimensional equations presented in this section are simplified to the case of one-dimensional isothermal flow. It should be noted that for the case of high-speed air propagation, the assumption of an adiabatic process is more realistic than an isothermal process. Nonetheless, the actual process is neither of these types. In fact, the actual process is polytropic. However, in this investigation, the isothermal flow assumption, employed in previous investigation, is used.[11]

Continuity equation

The general continuity equation for a fluid can be written as[10,24,25]

$$\int_V \frac{\partial \rho}{\partial t} \mathrm{d}V + \int_S \rho \mathbf{v} \bullet \mathbf{n} \mathrm{d}S = 0 \tag{1}$$

In this equation, V is the volume which is assumed to remain constant for the brake pipe, and therefore, no distinction is made between the volumes in the reference and current configurations; S is the surface area, ρ is the mass density, $\mathbf{v}$ is the velocity vector and $\mathbf{n}$ is the normal to the surface. Using the divergence theorem, the continuity equation can be written as

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{v}) = 0 \tag{2}$$

In this equation, ∇ is the divergence operator. Since air flow is considered in this study, the density ρ cannot be treated as a constant, and therefore, the assumption of incompressibility is not used in this study.

Momentum equation

The differential form of the momentum equation or the partial differential equation of equilibrium can be written in the following form[10]

$$\rho \mathbf{a} = (\nabla \boldsymbol{\sigma})^{\mathrm{T}} + \mathbf{f}_{\mathrm{b}} \tag{3}$$

In this equation, $\mathbf{a}$ is the acceleration vector, $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor and $\mathbf{f}_{\mathrm{b}}$ is the vector of body forces per unit volume. The first term of the preceding equation, which represents the inertia force, can be rewritten in a different form, which is more convenient to use when dealing with compressible fluids. Multiplying the continuity equation of equation (2) by the velocity vector $\mathbf{v}$, one obtains

$$\frac{\partial \rho}{\partial t} \mathbf{v} + \nabla(\rho \mathbf{v}) \mathbf{v} = \mathbf{0} \tag{4}$$

Using the expression for the total derivative of the velocity vector $\mathbf{v}$, the inertia term $\rho \mathbf{a}$ can be written as

$$\rho \mathbf{a} = \rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\nabla \mathbf{v})\mathbf{v} \tag{5}$$

Substituting equation (5) into equation (3) and adding equation (4), one obtains

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + (\rho(\nabla \mathbf{v}) + \nabla(\rho \mathbf{v}))\mathbf{v} = (\nabla \boldsymbol{\sigma})^{\mathrm{T}} + \mathbf{f}_{\mathrm{b}} \tag{6}$$

This equation is an alternate form of the momentum equation of equation (3).

Navier–Stokes equations

In order to obtain the fluid equations of motion, the constitutive equations of isotropic fluids are combined with the partial differential equations of equilibrium of equation (6). This leads to

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+(\rho(\nabla\mathbf{v})+\nabla(\rho\mathbf{v}))\mathbf{v}=\left\{-\nabla(p\mathbf{I})+\nabla(\lambda\mathrm{tr}(\mathbf{D})\mathbf{I})+\nabla(2\mu\mathbf{D})\right\}^{\mathrm{T}}+\mathbf{f}_{\mathrm{b}} \tag{7}$$

where p is the hydrostatic pressure, ρ is the mass density, T is the temperature, $\mathbf{D}$ is the rate of deformation tensor, and λ and μ are viscosity coefficients that depend on the fluid density and temperature. The preceding equation represents the general three-dimensional partial differential equations of motion for isotropic fluids. If the fluid is assumed to be Newtonian the preceding equation reduces to

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+(\rho(\nabla\mathbf{v})+\nabla(\rho\mathbf{v}))\mathbf{v}=\left\{-\nabla(p\mathbf{I})+\lambda\nabla(\mathrm{tr}(\mathbf{D})\mathbf{I})+2\mu\nabla(\mathbf{D})\right\}^{\mathrm{T}}+\mathbf{f}_{\mathrm{b}} \tag{8}$$

One-dimensional model

The assumption of one-dimensional air flow used in this study implies that the flow, at any cross-section, has only one direction along the longitudinal axis of the pipe, that is, the velocity components in the other directions are not considered. Furthermore, the magnitude of the flow velocity is assumed to be uniform at any cross-section. Consequently, shear stresses are neglected, and as a result, the off-diagonal elements of the Cauchy stress tensor are assumed to be zero.

In the case of one-dimensional air flow, the continuity equation of equation (2) reduces to

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+L=0 \tag{9}$$

In this equation, x is the longitudinal spatial pipe coordinate, L is the air leakage and u is the velocity component along the longitudinal x-coordinate of the brake pipe. Note that the preceding equation is a partial differential equation that depends on both time t and the spatial coordinate x. The effect of air flowing through the pipe branches can also be introduced systematically to the continuity equation in order to account for the mass flow rate. In the case of multiple branches connected to the main air pipe, a term can be added to the continuity equation as discussed in the companion paper in which the car control unit model is developed.[26]

In the case of one-dimensional inviscid flow, the Navier–Stokes equation of equation (8) becomes

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial\left(\rho u^2\right)}{\partial x}=-\frac{\partial p}{\partial x}+f_{\mathrm{b}} \tag{10}$$

Using the assumption of isothermal flow, one has the following relationship[25]

$$\frac{p}{\rho}=R_{\mathrm{g}}\Theta \tag{11}$$

In this equation, R_{g} is the gas constant which has units J/(kgK), and Θ is the local temperature (K). The relationship in equation (11) can be used to eliminate the air density ρ from the continuity and momentum equations leading to the following system of pressure/velocity coupled equations

$$\left.\begin{aligned}&\frac{\partial p}{\partial t}+\frac{\partial(pu)}{\partial x}+\gamma_t L=0\\&\frac{\partial(pu)}{\partial t}+\frac{\partial\left(pu^2\right)}{\partial x}+\gamma_t\frac{\partial p}{\partial x}=\gamma_t f_{\mathrm{b}}\end{aligned}\right\} \tag{12}$$

In this equation, $\gamma_t=R_{\mathrm{g}}\Theta$. Given the boundary and initial conditions, the preceding system of coupled partial differential equations can be solved for the pressure and velocity distributions using numerical methods as discussed in the following section.

Finite element formulation

In this study, a finite element procedure is used to transform the partial differential equations of equation (12) into a set of coupled first-order ordinary differential equations. These ordinary differential equations can be solved using the method of numerical integration to determine the pressure and velocity for different braking scenarios. It is worth mentioning that an approach based on the finite difference method can also be employed to solve the ordinary differential equations of the air flow in the brake pipe. However, previous studies have shown that the finite element formulation leads to a more accurate solution than the finite difference method.[11] Moreover, in the finite element formulation, one can use different element types and higher orders of interpolation.

Let $q=pu$ be a new variable. Using this definition, equation (12) can be rewritten as

$$\left.\begin{aligned}&\frac{\partial p}{\partial t}+\frac{\partial q}{\partial x}=-\gamma_t L\\&\frac{\partial q}{\partial t}+\frac{\partial(qu)}{\partial x}+\gamma_t\frac{\partial p}{\partial x}=\gamma_t f_{\mathrm{b}}\end{aligned}\right\} \tag{13}$$

In the finite element analysis, the brake pipe is assumed to consist of m finite elements. The domain of the element is defined by the spatial coordinate $x=x^e$, $0<x^e<l^e$, where l^e is the length of the finite element. Over the domain of the finite element, the variables p and q are interpolated using the following field

$$p^e(x,t)=\mathbf{S}_p^e\mathbf{p}^e,\quad q^e(x,t)=\mathbf{S}_q^e\mathbf{q}^e,\quad e=1,2,\ldots,m \tag{14}$$

where $\mathbf{S}_p^e$ and $\mathbf{S}_q^e$ are appropriate shape functions, and $\mathbf{p}^e$ and $\mathbf{q}^e$ are the vectors of nodal coordinates. Multiplying the first equation in equation (13) by the virtual change δp^e and the second equation by the virtual change δq^e, integrating over the volume, using the relationship $\mathrm{d}V^e = A^e \mathrm{d}x^e$, where A^e is the cross-sectional area; and using equation (14); one obtains the following system of first-order ordinary differential equations for the finite element e

$$\mathbf{M}^e \dot{\mathbf{e}}^e = \mathbf{Q}^e, \quad e = 1, 2, \ldots, m \tag{15}$$

In this equation,

$$\mathbf{e}^e = \begin{bmatrix} \mathbf{p}^e \\ \mathbf{q}^e \end{bmatrix}, \quad \mathbf{Q}^e = \begin{bmatrix} \mathbf{Q}_p^e \\ \mathbf{Q}_q^e \end{bmatrix}, \quad \mathbf{M}^e = \begin{bmatrix} \mathbf{M}_{pp}^e & \mathbf{M}_{pq}^e \\ \mathbf{M}_{qp}^e & \mathbf{M}_{qq}^e \end{bmatrix} \tag{16}$$

where

$$\left.\begin{aligned}
\mathbf{M}_{pp}^e &= \int_0^{l^e} A^e \mathbf{S}_p^{e\,\mathrm{T}} \mathbf{S}_p^e \mathrm{d}x, \ \mathbf{M}_{pq}^e = \mathbf{M}_{qp}^e = \mathbf{0}, \\
\mathbf{M}_{qq}^e &= \int_0^{l^e} A^e \mathbf{S}_q^{e\,\mathrm{T}} \mathbf{S}_q^e \mathrm{d}x \\
\mathbf{Q}_p^e &= -\left(\left(\int_0^{l^e} A^e \mathbf{S}_p^{e\,\mathrm{T}} \frac{\partial \mathbf{S}_q^e}{\partial x} \mathrm{d}x\right)\mathbf{q}^e + \int_0^{l^e} A^e \mathbf{S}_p^{e\,\mathrm{T}} \mathrm{d}x \gamma_t^e L^e\right) \\
\mathbf{Q}_q^e &= -\left(\int_0^{l^e} A^e \mathbf{S}_q^{e\,\mathrm{T}} \frac{\partial \mathbf{S}_q^e}{\partial x} u^e \mathrm{d}x\right)\mathbf{q}^e - \left(\int_0^{l^e} A^e \mathbf{S}_q^{e\,\mathrm{T}} \mathbf{S}_q^e \frac{\partial u^e}{\partial x} \mathrm{d}x\right)\mathbf{q}^e \\
&\quad -\gamma_t \left(\int_0^{l^e} A^e \mathbf{S}_q^{e\,\mathrm{T}} \frac{\partial \mathbf{S}_p^e}{\partial x} \mathrm{d}x\right)\mathbf{p}^e + \int_0^{l^e} A^e \mathbf{S}_q^{e\,\mathrm{T}} \mathrm{d}x \gamma_t f_\mathrm{b}
\end{aligned}\right\} \tag{17}$$

The finite element equations of equation (15) can be assembled using a standard finite element assembly procedure. This leads to the first-order ordinary differential equations of the brake pipe system which can be written in the following matrix form

$$\mathbf{M}\dot{\mathbf{e}} = \mathbf{Q} \tag{18}$$

In this equation, $\mathbf{e}$ is the vector of nodal coordinates, $\mathbf{M}$ is the brake pipe's global coefficient matrix that results from assembling the $\mathbf{M}^e$ element matrices and $\mathbf{Q}$ is the right-hand side vector that results from the assembly of the $\mathbf{Q}^e$ element vectors. Using the position of the handle of the automatic brake valve, the initial conditions and inputs for equation (18) can be defined and used with numerical integration methods to solve for the pressure and the velocity distribution. Using the approach described in this section in which the new variable $q = pu$ is introduced, one obtains constant symmetric $\mathbf{M}^e$ and $\mathbf{M}$ matrices. Therefore, one needs to define the **LU** factors of $\mathbf{M}$ only once at the start of the simulation. The effect of the air leakage in the finite element formulation presented in this section can be considered by introducing this effect at the nodal points using the *isotropic approach* or the *average density approach*.[11] The latter is the approach adopted in this investigation using a general finite element formulation that can be applied to different element types. However, in this study, linear finite elements are used to model the air flow in the brake pipe. Furthermore, although a train's brake pipe is often very long, it can be shown that relatively larger elements can be used to accurately model the air flow along the brake pipe. This is demonstrated in the numerical results that are presented in the companion paper.[26]

Another alternate approach to equation (13) is to use the same assumptions as well as equation (11) ($\rho = p/\gamma_t$) and combine the continuity equation (9) with the one-dimensional form of the momentum equation of equation (3) to obtain the following coupled system of first-order partial differential equations

$$\left.\begin{aligned}
&\frac{\partial p}{\partial t} + \frac{\partial (pu)}{\partial x} + \gamma_t L = 0 \\
&p\frac{\partial u}{\partial t} + pu\frac{\partial u}{\partial x} + \gamma_t \frac{\partial p}{\partial x} = \gamma_t f_\mathrm{b}
\end{aligned}\right\} \tag{19}$$

As an alternative to introducing the variable $q = pu$, one can use p and u instead of using p and q. One can, however, show that the use of equation (19) instead of equation (13) will not lead to a constant coefficient matrix.

Wall friction forces

A simple expression for the pipe's wall friction force F_S is used in this study. In the case of duct flow, one may assume that the force F_S is only related to the wall shear stress as[11]

$$F_\mathrm{S} = \tau \pi d \tag{20}$$

where d is the hydraulic diameter, and τ is the one-dimensional flow shear stress. They are defined as[11]

$$d = \sqrt{\frac{4}{\pi} A_\mathrm{avg}} = \sqrt{\frac{4}{\pi \Delta x} \int_x^{x+\Delta x} A \mathrm{d}x}, \quad \tau = f_\mathrm{w} \frac{\rho u^2}{8}\left(\frac{u}{|u|}\right) \tag{21}$$

where A_avg is the average area, and the wall friction factor f_w is a function of the local Reynolds number

Re. The following relation between f_w and Re is often used

$$f_w = a(Re)^b \tag{22}$$

where a and b are selected to give a good fit to data for different flow regimes; laminar, transition and turbulent. The expression for the friction force F_S can be introduced to the momentum equation. If the area of the pipe is constant, then $A_{avg} = A =$ constant.

Non-linear dynamic equations for a train

The mathematical models of the air flow in the brake pipe and the commands of the automatic brake valve can be used to define the braking scenarios that affect the train's longitudinal dynamics. In this section, the non-linear dynamic equations of the train cars are developed using trajectory coordinates. It is assumed that rail vehicle dynamics has no effect on the air flow in the brake pipe, whereas the braking forces can have a significant effect on the train's longitudinal forces.

Position, velocity and acceleration

In order to develop the non-linear dynamic equations of motion of the train, the global position vector of an arbitrary point on a car body is first defined. The position vector $\mathbf{r}^i$ of an arbitrary point on body i with respect to the global coordinate system can be defined as shown in Figure 2 as[27]

$$\mathbf{r}^i = \mathbf{R}^i + \mathbf{A}^i\bar{\mathbf{u}}^i \tag{23}$$

where $\mathbf{R}^i$ is the global position vector of the origin of the body coordinate system, $\bar{\mathbf{u}}^i$ is the position vector of the arbitrary point on the body with respect to the local coordinate system and $\mathbf{A}^i$ is the rotation matrix that defines the orientation of the local coordinate system with respect to the global system. In rigid-body dynamics, $\bar{\mathbf{u}}^i$ is constant and does not depend on time. Differentiating equation (23) with respect to time, one obtains the absolute velocity vector defined as

$$\dot{\mathbf{r}}^i = \dot{\mathbf{R}}^i + \boldsymbol{\omega}^i \times \mathbf{u}^i \tag{24}$$

where $\boldsymbol{\omega}^i$ is the absolute angular velocity vector defined in the global coordinate system, and $\mathbf{u}^i = \mathbf{A}^i\bar{\mathbf{u}}^i$. The absolute acceleration vector is obtained by differentiating the preceding equation with respect to time, leading to

$$\ddot{\mathbf{r}}^i = \ddot{\mathbf{R}}^i + \boldsymbol{\alpha}^i \times \mathbf{u}^i + \boldsymbol{\omega}^i \times \left(\boldsymbol{\omega}^i \times \mathbf{u}^i\right) \tag{25}$$

where α^i is the angular acceleration vector of body i. The preceding equation can also be written in the following alternate form

$$\ddot{\mathbf{r}}^i = \ddot{\mathbf{R}}^i + \mathbf{A}^i\left(\bar{\boldsymbol{\alpha}}^i \times \bar{\mathbf{u}}^i + \bar{\boldsymbol{\omega}}^i \times \left(\bar{\boldsymbol{\omega}}^i \times \bar{\mathbf{u}}^i\right)\right) \tag{26}$$

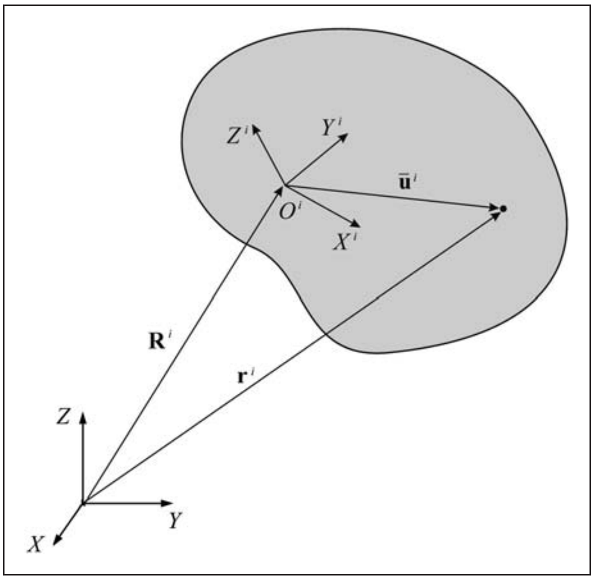

Figure 2. Coordinate systems.[23]

where $\boldsymbol{\alpha}^i = \mathbf{A}^i\bar{\boldsymbol{\alpha}}^i$ and $\boldsymbol{\omega}^i = \mathbf{A}^i\bar{\boldsymbol{\omega}}^i$. The angular velocity vectors defined, respectively, in the global and body coordinate systems can be written in terms of the time derivatives of the orientation coordinates $\boldsymbol{\theta}^i$ as follows

$$\boldsymbol{\omega}^i = \mathbf{G}^i\dot{\boldsymbol{\theta}}^i, \quad \bar{\boldsymbol{\omega}}^i = \bar{\mathbf{G}}^i\dot{\boldsymbol{\theta}}^i \tag{27}$$

where $\mathbf{G}^i$ and $\bar{\mathbf{G}}^i$ can be expressed in terms of the orientation parameters $\boldsymbol{\theta}^i$.[23]

Trajectory coordinates

The trajectory coordinate formulation is suited for the study of the train's longitudinal force dynamics since the degrees of freedom of the car body can be systematically reduced to a set that can be related to the track geometry. A centroidal body coordinate system is introduced for each of the railroad vehicle components. In addition to the centroidal body coordinate system, a *body/track coordinate system* that follows the motion of the body is introduced. The location of the origin and the orientation of the body/track coordinate system are defined using one geometric parameter s^i that defines the distance traveled by the body along the track. The body coordinate system is selected such that it has no displacement in the longitudinal direction of motion with respect to the body/track coordinate system. Two translational coordinates, y^{ir} and z^{ir}; and three angles, ψ^{ir}, ϕ^{ir} and θ^{ir}, are used to define the position and orientation of the body coordinate system with respect to the body/track coordinate system $X^{ti}Y^{ti}Z^{ti}$, as shown in Figure 3. Therefore, for each body i in the system, the following six trajectory coordinates can be used:

$$\mathbf{p}^i = [\,s^i \quad y^{ir} \quad z^{ir} \quad \psi^{ir} \quad \phi^{ir} \quad \theta^{ir}\,]^{\mathrm{T}} \tag{28}$$

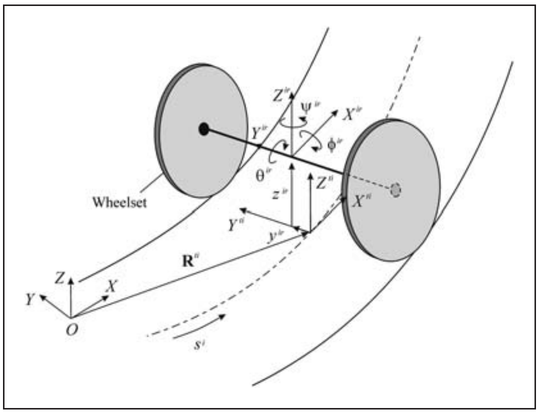

Figure 3. Trajectory coordinates.[23]

In terms of these coordinates, the global position vector of the center of mass of body i can be written as

$$\mathbf{R}^i = \mathbf{R}^{ti} + \mathbf{A}^{ti}\bar{\mathbf{u}}^{ir} \tag{29}$$

where $\bar{\mathbf{u}}^{ir}$ is the position vector of the center of mass with respect to the body/track coordinate system, $\mathbf{R}^{ti}$ is the global position vector of the origin of the trajectory coordinate system, and $\mathbf{A}^{ti}$ is the matrix that defines the orientation of the body/track coordinate system and is a function of three predefined Euler angles ψ^{ti}, ϕ^{ti} and θ^{ti} which are used to define the track geometry. The vector $\mathbf{R}^{ti}$ and the matrix $\mathbf{A}^{ti}$ are functions of only one time-dependent arc length parameter s^i. For a given s^i, one can also determine the three Euler angles $\boldsymbol{\theta}^{ti}(s^i) = [\,\psi^{ti}(s^i) \quad \theta^{ti}(s^i) \quad \phi^{ti}(s^i)]^{\mathrm{T}}$ that enter into the formulation of the rotation matrix $\mathbf{A}^{ti}$.[23] The vector $\bar{\mathbf{u}}^{ri}$ can be written as

$$\bar{\mathbf{u}}^{ir} = [\,0 \quad y^{ir} \quad z^{ir}\,]^{\mathrm{T}} \tag{30}$$

The matrix $\mathbf{A}^{ir}$ that defines the orientation of the body coordinate system with respect to the body/track coordinate system can be expressed in terms of the three time-dependent Euler angles $\boldsymbol{\theta}^{ir} = [\,\psi^{ir} \quad \theta^{ir} \quad \phi^{ir}\,]^{\mathrm{T}}$ previously defined.

Equations of motion

A velocity transformation matrix that relates the absolute Cartesian accelerations to the trajectory coordinate accelerations can be systematically developed. Using the velocity transformation and the Newton–Euler equations that govern the spatial motion of the rigid bodies, the equations of motion of the car bodies expressed in terms of the trajectory coordinates can be developed. The following form of the Newton–Euler equations is used in this investigation[27]

$$\begin{bmatrix} m^i\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{I}}^i_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \ddot{\mathbf{R}}^i \\ \bar{\boldsymbol{\alpha}}^i \end{bmatrix} = \begin{bmatrix} \mathbf{F}^i_{\mathrm{e}} \\ \bar{\mathbf{M}}^i_{\mathrm{e}} - \bar{\boldsymbol{\omega}}^i \times \left(\bar{\mathbf{I}}^i_{\theta\theta}\bar{\boldsymbol{\omega}}^i\right) \end{bmatrix} \tag{31}$$

where m^i is the mass of the rigid body; $\mathbf{I}$ is a 3×3 identity matrix; $\bar{\mathbf{I}}^i_{\theta\theta}$ is the inertial tensor defined with respect to the body coordinate system; $\mathbf{F}^i_{\mathrm{e}}$ is the resultant of the external forces applied on the body defined in the global coordinate system; and $\bar{\mathbf{M}}^i_{\mathrm{e}}$ is the resultant of the external moments acting on the body defined in the body coordinate system. The forces and moments acting on the body include the effect of gravity, braking forces, coupler forces, and tractive effort and motion resisting forces. Coupler forces, in particular, have a significant effect on longitudinal train dynamics, and therefore, accurate computer models must be developed for these forces.[8,28]

If $\mathbf{a}^i$ is the vector of absolute Cartesian accelerations of the body, one can use the kinematic description given in this section to write the Cartesian accelerations in terms of the trajectory accelerations as

$$\mathbf{a}^i = \mathbf{B}^i\ddot{\mathbf{p}}^i + \gamma^i \tag{32}$$

In this equation, $\mathbf{a}^i = \left[\,\ddot{\mathbf{R}}^{i\mathrm{T}} \quad \bar{\boldsymbol{\alpha}}^{i\mathrm{T}}\,\right]^{\mathrm{T}}$, $\mathbf{B}^i$ is a velocity transformation matrix and γ^i is a quadratic velocity vector.[23] Substituting equation (32) into equation (31) and pre-multiplying by the transpose of the velocity transformation matrix $\mathbf{B}^i$, one obtains the dynamic equations expressed in terms of the trajectory coordinates, as described in detail in Shabana et al.[23]

Automatic brake valve model

The air pressure in the brake pipe system is controlled by the automatic brake valve. The mathematical model of the 26C automatic brake valve developed by Abdol-Hamid,[11] is used in this investigation. The valve and its main components are shown in Figure 4. The primary function of the automatic brake valve is to control the air pressure in the brake pipe allowing for the application or release of the train and locomotive brakes. By changing the position of the handle of this valve, the air pressure can be reduced (venting air to the atmosphere) or increased (recharging) at a controlled rate to *apply* or *release* the brakes, respectively. The main components of the 26C automatic brake valve are the *regulating valve*, *relay valve*, *brake pipe cut-off valve*, *vent* and *emergency valves*, and *suppression valve*. The main function of the regulating valve is to control the pressure in the *equalizing reservoir* which has a value that depends on the position assumed by the handle of the automatic brake valve. This pressure controls the relay valve which controls the air pressure along the brake pipe for the purpose of brake application or release. The brake pipe cut-off valve is located between the relay valve and the brake pipe, and its function is to allow communication with the pipe only when a threshold pressure value is reached. The vent and emergency valves are activated only in case of emergency and their function is to vent the air in both the brake pipe and the

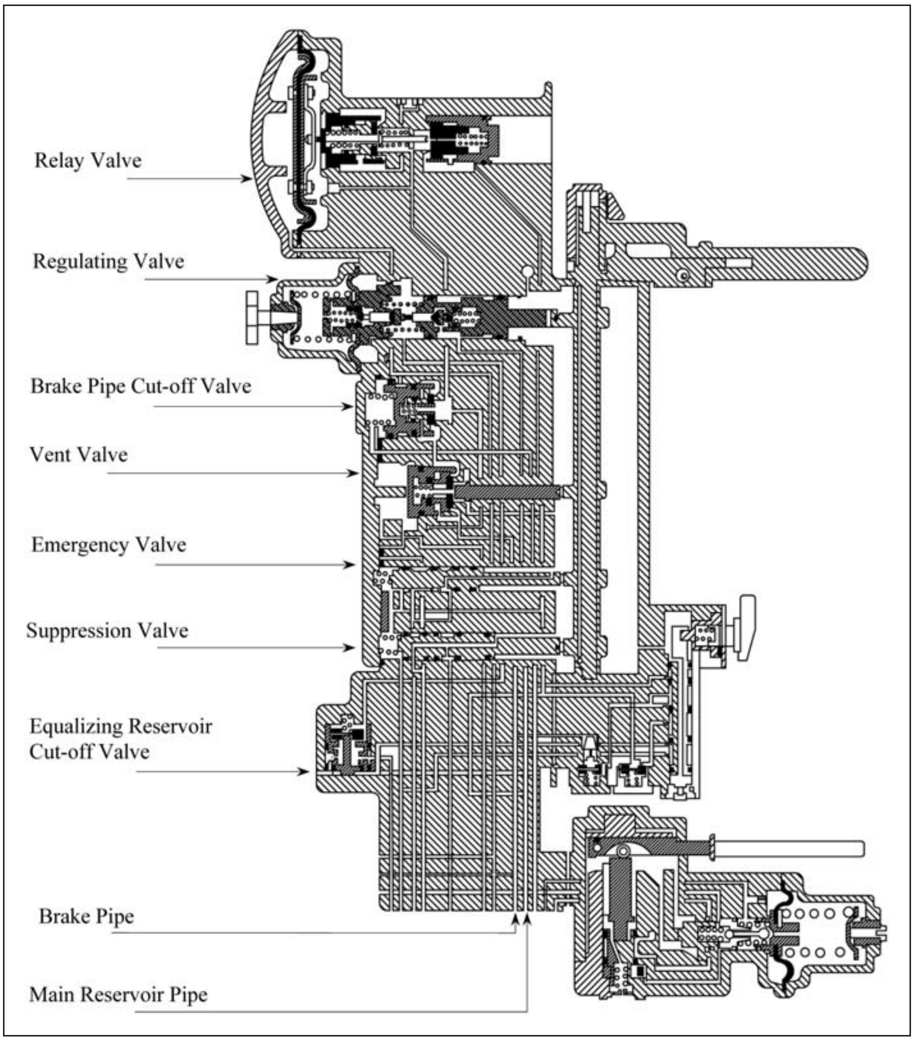

Figure 4. A schematic diagram of a 26C valve.

equalizing reservoir to the atmosphere. The resulting sudden drop in the air pressure leads to a faster brake application. The suppression valve is used to control the communication between the equalizing and the main reservoirs. In this investigation, only the mathematical models of the first three valves; relay valve, regulating valve and brake pipe cut-off valve, are developed. As in Abdol-Hamid,[11] a variable or a component $y_{i,j}$ of the automatic brake valve denotes variable or component number j associated with valve i; for the relay valve, $i = 1$; for the regulating valve, $i = 2$; and for the brake pipe cut-off valve, $i = 3$.

In order to be able to develop the valve equations, several basic thermodynamics relationships must be used.[24,25] The first is the universal law of gases for a volume V_f of a component f given by

$$P_f V_f = m_f R_g \Theta_f \tag{33}$$

where P_f is the absolute pressure inside the volume (N/m^2), V_f is the volume (m^3), m_f is the mass (kg), R_g is the gas constant (J/(kg K)) and Θ_f is the temperature (K). Differentiating the preceding equation with respect to time and assuming isothermal process ($\Theta_f = \Theta$ is constant), one obtains

$$\frac{\mathrm{d}P_f}{\mathrm{d}t} = \frac{1}{V_f}\left(R_g \Theta \frac{\mathrm{d}m_f}{\mathrm{d}t} - P_f \frac{\mathrm{d}V_f}{\mathrm{d}t}\right) \tag{34}$$

There are different formulas for calculating the rate of mass flow through orifices.[11,24,25] In this investigation, the following equation is used for air

$$\dot{m}_{\mathrm{air}} = 0.6 A P_{\mathrm{d}} \sqrt{\frac{|r^2 - 1|}{R_g \Theta}} \frac{|r - 1|}{r - 1} \tag{35}$$

In this equation, A is the area, $r = P_{\mathrm{u}}/P_{\mathrm{d}}$, and P_{u} and P_{d} are, respectively, the pressure upstream and downstream of the orifice. As reported in Abdol-Hamid,[11] the difference between different formulas is always less than 10%, which can be less than the error in calculating the geometric area of the orifice. The preceding equation, however, is simpler to use because it is valid for any value of the pressure ratio r (sonic or subsonic), while using other formulas requires a check of the value of r against the critical pressure ratio

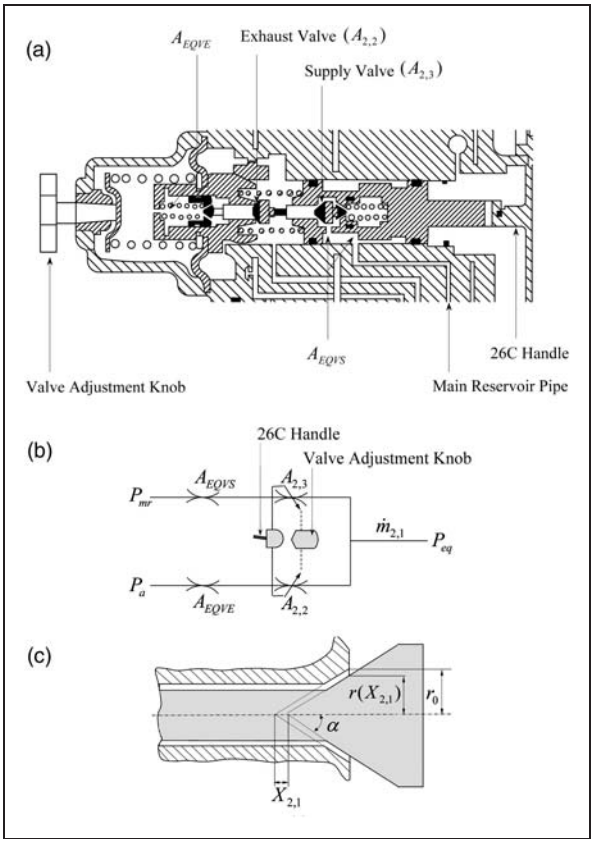

Figure 5. A 26C regulating valve[11] (a) schematic diagram of the valve, (b) fluid network and (c) valve areas.

$r_c = 1.893$ that defines whether the flow is sonic or subsonic.

Regulating valve operation

The main function of the regulating valve, shown in Figure 5(a), is to control the equalizing reservoir pressure which controls the operation of the relay valve that regulates the air pressure in the brake pipes. The regulating valve is controlled by one cam mounted on the shaft of the handle of the automatic brake valve. The valve has a self-lapping feature that provides an automatic control of the equalizing reservoir pressure P_{eq} against leakage or overcharge.[11]

Brake application and release

When the handle of the automatic brake valve is in the r*elease/recharge position*, the air pressure of the brake pipe P_{bp} must increase and the equalizing reservoir must be recharged. In this scenario, the cam rotates to a higher position allowing the supply valve seat $A_{2,3}$ to move left away from the handle, and causing the exhaust valve $A_{2,2}$ to be sealed. As a result, the air from the main reservoir with pressure P_{mr} flows through the supply valve $A_{2,3}$ and supply orifice A_{EQVS} reaching the inner diaphragm chamber $V_{2,1}$, through the equalizing reservoir cut-off valve to, finally, the equalizing reservoir V_{eq}. The equalizing reservoir pressure P_{eq} increases causing the supply valve to start closing.

When the handle is within the *service mode sector*, the air pressure of the brake pipe P_{bp} has to decrease in order to apply the car brakes. The regulating valve has to reduce the equalizing reservoir pressure P_{eq} which controls the relay valve that regulates the air flow to and from the brake pipe. The pressure reduction depends on the position of the handle of the automatic brake valve relative to the full service position. In the case of brake application, the cam rotates to a lower position allowing the regulating valve spool to move towards the handle (right), while the supply valve $A_{2,3}$ is sealed and the exhaust valve $A_{2,2}$ is opened. At the same time, the air pressure P_{mr} from the main reservoir is removed, causing the equalizing reservoir cut-off valve to close. This allows the air to flow only from the equalizing reservoir to the regulating valve, preventing any possible increase of the equalizing reservoir pressure P_{eq} during the entire application. Thus, the air flows from the equalizing reservoir V_{eq} through the cut-off valve, the inner diaphragm chamber $V_{2,1}$, the exhaust valve $A_{2,2}$ and its orifice A_{EQVE} to reach finally the atmosphere. As the pressure in the equalizing reservoir P_{eq} decreases, the exhaust valve seat moves right, causing the exhaust valve to start closing. The fluid network diagram of the regulating valve is shown in Figure 5(b). From this figure, one can see the two orifice areas (A_{EQVS} and A_{EQVE}), which are fixed, and the $A_{2,2}$ and $A_{2,3}$ areas of the exhaust and supply valve, respectively, which are variable and controlled by the handle position and air flow. The equalizing reservoir is connected to the outer chamber of the relay valve, and therefore, the pressure $P_{1,1}$ inside the relay valve's outer chamber can be controlled by the equalizing reservoir pressure P_{eq}. The pressure $P_{1,1}$ is used to control the operations of the relay valve which controls the pressure in the brake pipe. The operation of the relay valve is discussed in more detail in the section 'Relay valve operation'.

Pressure rate

In order to evaluate the rate of change of the pressure in the outer chamber of the relay valve $P_{1,1}$, the mass flow rate $\dot{m}_{1,1}$ through the feedback orifice $A_{1,1}$ must be calculated. The rate of change of equalizing reservoir pressure P_{eq} depends on the rate of the mass flow $\dot{m}_{1,1}$ to the relay valve through $A_{1,1}$, plus the mass flow rate $\dot{m}_{2,1}$ through either the regulating supply valve or the exhaust valve depending on the braking scenario. Let P_{REG} denote the pressure inside the regulating valve, and A_{REG} denote the equivalent areas for both the supply and the exhaust valve connections. The equivalent area for the supply valve used in the case of brake release can be obtained assuming series connection as

$$A_{\mathrm{REG}} = \frac{1}{\sqrt{\left(1/A_{\mathrm{EQVS}}^2\right) + \left(1/A_{2,3}^2\right)}} = \frac{A_{\mathrm{EQVS}}A_{2,3}}{\sqrt{A_{\mathrm{EQVS}}^2 + A_{2,3}^2}} \tag{36}$$

where $A_{2,3}$ is variable depending on the handle position, while A_{EQVS} is constant (see Appendix). In the case of brake application, the exhaust valve is used. In this case, the equivalent area for this valve connection is given, assuming again a series connection, as

$$A_{\mathrm{REG}} = \frac{1}{\sqrt{\left(1/A_{\mathrm{EQVE}}^2\right) + \left(1/A_{2,2}^2\right)}} = \frac{A_{\mathrm{EQVE}}A_{2,2}}{\sqrt{A_{\mathrm{EQVE}}^2 + A_{2,2}^2}} \tag{37}$$

In this equation, $A_{2,2}$ can vary depending on the handle position; while A_{EQVE} is constant.

Applying equation (34) to the rate of change of P_{eq} and assuming the volume V_{eq} to be constant, one obtains

$$\frac{\mathrm{d}P_{\mathrm{eq}}}{\mathrm{d}t} = \frac{R_{\mathrm{g}}}{V_{\mathrm{eq}}}\left(\frac{\mathrm{d}m_{2,1}}{\mathrm{d}t} - \frac{\mathrm{d}m_{1,1}}{\mathrm{d}t}\right) \tag{38}$$

Using equation (35), the mass flow rate $\dot{m}_{2,1}$ can be written as

$$\dot{m}_{2,1} = 0.6A_{\mathrm{REG}}P_{\mathrm{eq}}\sqrt{\frac{\left|r^2-1\right|}{R_{\mathrm{g}}\Theta}}\frac{|r-1|}{r-1} \tag{39}$$

where $r = P_{\mathrm{REG}}/P_{\mathrm{eq}}$ and $P_{\mathrm{REG}} = P_{\mathrm{mr}}$ in the case of brake release and $P_{\mathrm{REG}} = P_{\mathrm{a}}$ in the case of brake application, where P_{a} is the atmospheric pressure.

Relay valve operation

Figure 6(a) shows a schematic diagram of the relay valve and its components. The diaphragm chamber is divided into two parts; the outer chamber with volume $V_{1,1}$ and pressure $P_{1,1}$, and the inner chamber with volume $V_{1,2}$ and pressure $P_{1,2}$. The displacement of the diaphragm causes the diaphragm rod to move, thereby controlling the relay supply valve $A_{1,6}$ and the relay exhaust valve $A_{1,4}$. As shown in Figure 6(b), there is always a gap X_0 between the diaphragm rod and one of the two relay valves (supply and exhaust); X_0 is the gap between the diaphragm rod and the supply valve when the diaphragm is in its rest position. The displacement of the diaphragm is measured by the variable $X_{1,1}$ along the longitudinal axis of the diaphragm rod; $X_{1,1} = 0$ in the diaphragm rest

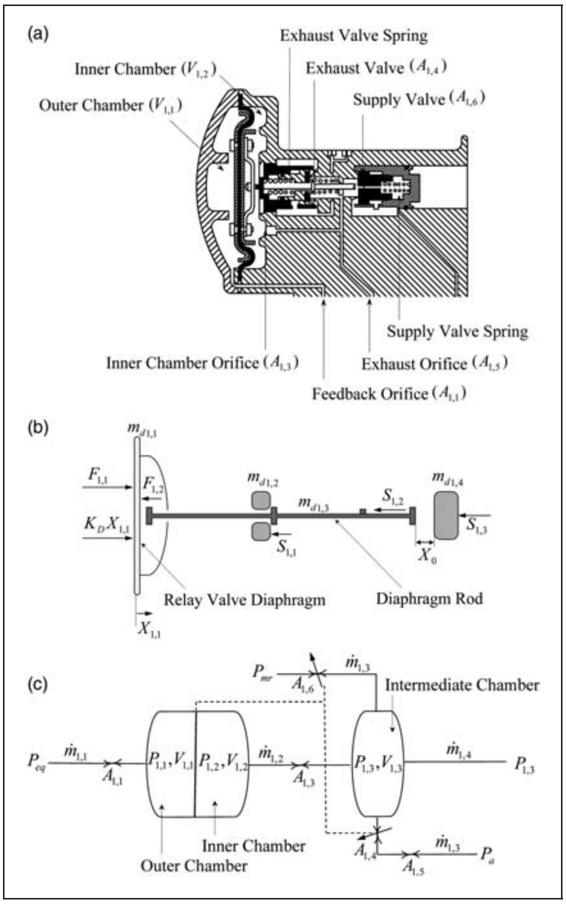

Figure 6. The relay valve[11] (a) schematic diagram of the valve, (b) free body diagram and (c) the fluid network.

position. When the diaphragm moves to the right, it pushes the rod which reaches the supply valve after a displacement X_0; when moving to the left, instead, the diaphragm is free until a displacement of $X_{1,1} = -X_{\mathrm{I}}$ where it starts to pull the rod and open the exhaust valve. Moving right opens the supply valve $A_{1,6}$ (fully open when the maximum right displacement, controlled by stops, $X_{1,1} = X_{\mathrm{S}}$ is reached); and moving left controls the opening of the exhaust valve $A_{1,4}$ (fully open when the maximum left displacement, controlled by stops, $X_{1,1} = -X_{\mathrm{E}}$ is reached). Because the gap is always present between the rod and at least one of the valves, only one (or none) of the two valves can be open at the same time. Through the feedback orifice $A_{1,1}$ of the equalizing reservoir, the air can enter or leave the outer chamber $V_{1,1}$, adjusting the value of the pressure $P_{1,1}$. This pressure exerts a force $F_{1,1}$ on the outer side $A_{1,2}$ of the diaphragm causing it to move. This force is opposed, at the other side of the diaphragm $A_{1,2}$, by the force $F_{1,2}$ exerted by the pressure $P_{1,2}$ of the inner chamber $V_{1,2}$. The other forces acting are those of the springs; the exhaust valve spring force $S_{1,1}$, the supply valve spring force $S_{1,3}$ and the diaphragm rod spring force $S_{1,2}$. The effect of the stiffness of the diaphragm can also be considered.

Brake release and application

During the *brake release/charge mode*, the equalizing reservoir pressure P_{eq} is increased by the action of the regulating valve, as previously explained. Consequently, $P_{1,1}$ also increases so that the net force acting on the diaphragm makes it move to the right causing the supply valve to start to open. The air, from the main reservoir with pressure P_{mr}, flows through the supply valve to the intermediate volume $V_{1,3}$ (located between the inner chamber $V_{1,2}$ and the brake pipe cut-off valve). If the air pressure at the brake pipe cut-off valve is large enough to overcome the valve spring preload, the valve opens and the air can flow to the brake pipe, increasing its pressure P_{bp} causing the brakes to release. The pressure on both sides of the diaphragm ($P_{1,1}$ of the outer chamber and $P_{1,2}$ of the inner chamber) increases with almost the same rate. When the brake pipe cut-off valve opens, $P_{1,2}$ in the inner chamber drops: this causes the diaphragm rod to move to the right, increasing the opening of the supply valve, and causing more air to flow through this valve, compensating for the pressure drop. Slowly the brake pipe pressure P_{bp} and the pressure $P_{1,2}$ of the inner chamber reach the value of $P_{1,1}$ (very close to the value of P_{eq}) causing the diaphragm to move left towards its lap position ($X_{1,1} = X_0$) and the supply valve starts closing.

In the case of brake application, the regulating valve causes the equalizing reservoir pressure P_{eq} to decrease, and consequently, the value of the pressure $P_{1,1}$ in the outer chamber decreases. As a result, the net force on the diaphragm reverses its direction and the diaphragm and its rod starts moving to the left opening the exhaust valve. The air flows from the brake pipe through the exhaust valve opening $A_{1,4}$ and its orifice $A_{1,5}$ to the atmosphere. As a result, the brake pipe pressure P_{bp} drops at a service rate of application causing the brake application. In the steady state case, the pressures $P_{1,1}$, $P_{1,2}$ and P_{bp} are approximately equal to the equalizing reservoir pressure P_{eq}. In the case of an emergency braking procedure, the pressure in the brake pipe decreases rapidly (at an emergency rate of application) to the atmospheric pressure. To avoid the situation in which the relay valve goes into the release mode, the vent valve is used to vent the air in the equalizing reservoir V_{eq} to the atmosphere, causing the pressure P_{eq} to decrease and, consequently, the pressure $P_{1,1}$ also decreases.

Simplified model

In order to develop a more efficient computational relay valve model, several assumptions can be made.[11] First, the effect of the diaphragm's inertia forces $m_{d1}\left(\mathrm{d}u_{1,1}/\mathrm{d}t\right)$ can be neglected. Second, the outer chamber pressure $P_{1,1}$ can be assumed equal to the equalizing reservoir pressure P_{eq}, that is, $P_{1,1} \approx P_{eq}$. Third, the pressure $P_{1,3}$ in the intermediate chamber can be assumed equal to the inner chamber pressure $P_{1,2}$, that is, $P_{1,3} \approx P_{1,2}$. Fourth, the intermediate chamber volume $V_{1,3}$ is small and can be assumed equal to zero ($V_{1,3} \approx 0$). Using the first three assumptions, one can show that the displacement $X_{1,1}$ takes the following form

$$X_{1,1} = \begin{cases} X_S & H_0 < D_P \\ B_0(D_P - H_1) + X_0 & H_1 < D_P \leqslant H_0 \\ X_0 & H_2 < D_P \leqslant H_1 \\ B_1(D_P - H_3) & H_3 < D_P \leqslant H_2 \\ 0 & 0 < D_P \leqslant H_3 \\ B_2 D_P & H_4 < D_P \leqslant 0 \\ -X_I & H_5 < D_P \leqslant H_4 \\ B_3(D_P - H_5) - X_I & H_6 < D_P \leqslant H_5 \\ -X_E & H_6 > D_P \end{cases} \tag{40}$$

where

$$\left.\begin{aligned} B_0 &= \frac{A_{1,1}}{K_{1,2} + K_{1,3} + K_D}, & B_1 &= \frac{A_{1,2}}{K_{1,2} + K_D}, \\ B_2 &= \frac{A_{1,2}}{K_D}, & B_3 &= \frac{A_{1,2}}{K_{1,1} + K_{1,2} + K_D} \\ H_0 &= \frac{X_S - X_0}{B_0} + H_1, & H_1 &= \frac{L_{1,3}}{A_{1,2}} + H_2, \\ H_2 &= \frac{X_0}{B_1} + H_3, & H_3 &= \frac{L_{1,2}}{A_{1,2}} \\ H_4 &= -\frac{X_I}{B_2}, & H_5 &= \frac{L_{1,2} - L_{1,1}}{A_{1,2}} + H_4, \\ H_6 &= \frac{X_I - X_E}{B_3} + H_5, & D_P &= P_{eq} - P_{1,3} \end{aligned}\right\} \tag{41}$$

Note that $P_{1,3} = P_{bp}$ for $-X_I \leqslant X_{1,1} \leqslant X_0$. The nine cases presented in equation (40) correspond, respectively, to the scenarios of the supply valve fully open X_S, supply valve opening, X_0, both valves closed, diaphragm rest position, diaphragm free movement, start exhaust valve opening $-X_I$ and exhaust valve fully open $-X_E$. In order to determine $X_{1,1}$ of equation (40), one must determine $P_{1,3}$ which enters in the formulation of D_P. By using the assumption $\dot{m}_{1,3} = \dot{m}_{1,4}$, one can show that[11]

$$P_{1,3} = \sqrt{\frac{P_{bp}^2 A_{3,3}^2 + P^2 A^2}{A_{3,3}^2 + A^2}} \tag{42}$$

where A and P can be either $A_{1,6}$ and P_{mr} (for the supply valve) or A_{EX} and P_a (for the exhaust valve), respectively.

Brake pipe cut-off valve operation

The brake pipe cut-off valve is shown in Figure 7(a). The forces acting on the piston are the forces exerted

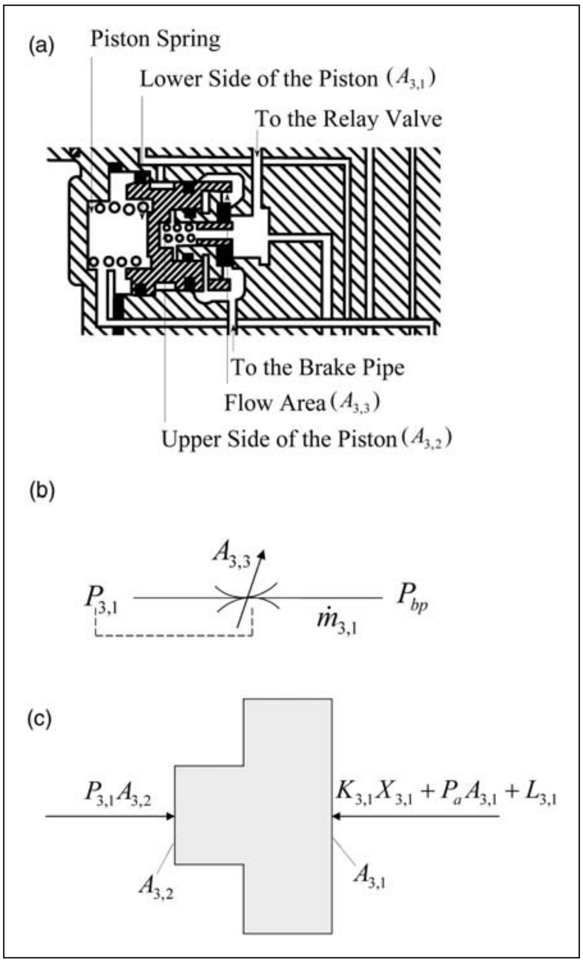

Figure 7. Brake pipe cut-off valve[11](a) schematic diagram of the valve, (b) the fluid network and (c) the free body diagram.

by the pressure $P_{3,1}$ ($P_{3,1} = P_{1,3}$) which act on the upper side of the piston $A_{3,2}$; the force exerted by the atmospheric pressure P_a which acts on the lower side $A_{3,1}$; and the spring force $S_{3,1} = K_{3,1}X_{3,1}$ (plus pre-load). The valve opens if $P_{3,1}$ overcomes the resultant of the pressure P_a force and the spring pre-load force. Figure 7(b) shows how the pressure $P_{3,1}$ coming from the relay valve through the orifice $A_{3,3}$, controls the opening of the valve, allowing or preventing the air flow to the brake pipe.

In order to calculate the mass flow rate through the brake pipe cut-off valve $\dot{m}_{3,1}$, the flow area $A_{3,3}$ needs to be evaluated. This orifice is controlled by the pressure $P_{3,1}(= P_{1,3})$ acting on the inner side of the piston $A_{3,2}$, which determines the piston and the valve displacement. The piston in the brake pipe cut-off valve has a very small mass which can be neglected. Using the valve free body diagram shown in Figure 7(c), the equilibrium equation is

$$P_{3,1}A_{3,2} - \left(P_aA_{3,1} + K_{3,1}X_{3,1} + L_{3,1}\right) = 0 \qquad (43)$$

where $K_{3,1}$ is the valve spring rate, $X_{3,1}$ is the piston displacement and $L_{3,1}$ is the spring pre-load. The valve opens when the force due to the pressure $P_{3,1}$ overcomes the forces generated by the atmospheric pressure P_a and the spring pre-load force $L_{3,1}$, that is, when $P_{3,1}A_{3,2} \geqslant P_aA_{3,1} + L_{3,1}$. This condition can be rewritten as $P_{3,1} \geqslant P_B$, where $P_B = \left(P_aA_{3,1} + L_{3,1}\right)/A_{3,2}$.

Note that the area $A_{3,3}$ of the annular ring created by the displacement $X_{3,1}$ and the diameter $D_{3,3}$ is $A_{3,3} = \pi D_{3,3}X_{3,1}$. In order to calculate this area, one needs to evaluate $X_{3,1}$ using equation (43) as

$$\left.\begin{array}{ll} X_{3,1} = 0 & P_{3,1} \leqslant P_B \\ X_{3,1} = A_{3,2}\dfrac{P_{3,1} - P_B}{K_{3,1}} & P_B \leqslant P_{3,1} < P_B + \varepsilon \\ X_{3,1} = X_B & P_{3,1} \geqslant P_B + \varepsilon \end{array}\right\} \qquad (44)$$

In this equation, $P_B + \varepsilon$ is the value of pressure that makes the piston reach the maximum displacement X_B (cut-off valve fully open). These two values depend on the dimensions of the brake pipe cut-off valve.

Integration of models of the automatic brake valve, brake pipe and the train's dynamics

The second-order non-linear dynamic equations for the train can be converted to a set of first- order ordinary differential equations that can be integrated using a standard numerical integration method. As shown in this paper, the first-order ordinary differential equations that govern the air flow in the brake pipe can be written in the form $\mathbf{M}\dot{\mathbf{e}} = \mathbf{Q}$ (see equation (18)). For a given value of the right-hand side vector $\mathbf{Q}$ and a set of initial conditions $\mathbf{e}_0$, the air flow in the differential equations for the brake pipe can be solved numerically with the differential equations for the train to determine the air pressure and density as a function of time. In this section, two different scenarios will be discussed in order to demonstrate the integration of the equations describing the air flow in the brake pipe and automatic brake valve with the non-linear dynamic equations for the train. The first scenario is a brake release after a train stop; whereas the second scenario is a brake application while the train is in motion. It should be noted that a complete air brake model must also include a CCU that will be discussed in the companion paper.[26]

Brake release

The steps of the algorithm used in this study for the brake release are summarized as follows.

1. Before the train starts moving, one has the following conditions: the air pressures in the brake pipe and equalizing reservoir are equal to the full

service brake pipe pressure, while the air flow velocity in the brake pipe is equal to zero; the brake pipe cut-off valve is closed; and the relay valve is in the *intermediate state*, that is, $P_{1,3} = P_{bp}$.

2. For a given set of initial conditions for the train car body, the train's dynamic equations are formulated as a set of first-order ordinary differential equations that can be solved using a standard numerical integration method. These equations are solved simultaneously with the air flow differential equations (equation (18)), the intermediate chamber pressure differential equation (equation (53)), the equalizing reservoir pressure differential equation (equation (38)) and the equations for the CCU presented in the companion paper for the given initial conditions defined in the previous step.[26]
3. For the case of a brake release, the handle of the automatic brake valve is moved to the release position causing the supply valve of the regulating valve to open. Depending on the value of X_{REG} as defined in the Appendix, equation (38) given by

$$\frac{dP_{eq}}{dt} = \frac{R_g}{V_{eq}}\left(\frac{dm_{2,1}}{dt} - \frac{dm_{1,1}}{dt}\right)$$

can be numerically solved with the train's dynamic equations. For the case of the simplified relay valve model, $\dot{m}_{1,1} = 0$, and $\dot{m}_{2,1}$ can be determined using equation (39) which is function of A_{REG} that depends on X_{REG} of equation (49). Therefore, equation (38) can be solved for the pressure in the equalizing reservoir P_{eq}. The current value of the pressure P_{eq} is monitored in order to check on the relay valve state.
4. As the equalizing reservoir pressure P_{eq} reaches a certain value, the relay supply valve starts opening. The supply valve's area which is a function of $D_P = P_{eq} - P_{1,3}$ is defined in equation (61) ($X_{1,1}$ that appear in this equation can be determined using equation (40)). The pressure $P_{1,3}$ of the intermediate chamber starts to increase due to the incoming flow of air from the main reservoir. If the simplified relay valve model is used, the pressure $P_{1,3}$ can be determined using the equation.

$$P_{1,3} = \sqrt{\left(P_{bp}^2 A_{3,3}^2 + P_{mr}^2 A_{1,6}^2\right)/\left(A_{3,3}^2 + A_{1,6}^2\right)}$$

At the same time, the brake pipe cut-off valve remains closed, and therefore, $A_{3,3} = 0$.
5. When the pressure $P_{1,3}$ becomes larger than the pressure P_B by the equation $P_B = \left(P_a A_{3,1} + L_{3,1}\right)/A_{3,2}$, the brake pipe cut-off valve starts opening, and the air starts to flow to the brake pipe. The area of the brake pipe cut-off valve $A_{3,3}$ can be determined using equation (62) in the Appendix. Using the current values of $P_{1,3}$ and $A_{3,3}$, the mass flow rate of equation (55) is used to determine $\dot{m}_{1,4}$, which is used in the air flow equations of the brake pipe, to close the brake pipe cut-off valve. Note that the procedure used in this algorithm is based on equation (42) and equation (61) in the Appendix. An alternate procedure can be developed based on equation (63) presented in the Appendix.
6. As the air pressure P_{bp} of the brake pipe increases, the brake shoes begin to separate from the wheel axles, until the brake release process is completed. The process, thus, ends with the relay supply valve closed (with the assumption of no leakage) and the brake pipe cut-off valve fully open. At this configuration, $P_{1,3} = P_{bp}$. Note that the value of the pressure $P_{1,2}$, which is equal to $P_{1,3}$ in the simplified model, that determines the closing of the supply valve is controlled by the regulating valve; as X_{REG} reaches a specified known value, the pressure P_{eq} of the equalizing reservoir increases and reaches its maximum value, the regulating supply valve closes, and no more air flows into the equalizing reservoir. Recall that the motion of the relay valve's diaphragm depends on the pressure $P_{1,1}$ of the relay valve's outer chamber which is connected to the equalizing reservoir which has the pressure P_{eq}. As $P_{1,1}$ reaches its maximum value, the relay supply valve closes; and the air no longer flows from the main reservoir to the intermediate chamber of the relay valve.
7. When the relay supply valve closes, $A_{1,6} = 0$, and equation (42) shows that $P_{1,3} = P_{bp}$. This means that the pressure in the intermediate chamber will assume the value of the pressure in the brake pipe.

Brake application

The steps of the algorithm for the case of service brake application can be summarized as follows.

1. For a given set of initial conditions for the train car body and track geometry, the train's nonlinear dynamic equations are formulated as a set of first-order ordinary differential equations that can be solved using a standard numerical integration method. These equations are solved simultaneously with the differential equations describing the air flow (equation (18)), the equalizing reservoir pressure differential equation (equation (38)), the intermediate chamber pressure differential equation (equation (53), and CCU's equations (discussed in the companion paper[26] for a given set of initial conditions. The dynamic simulation of the train subject to different forces continues until the brakes are applied.
2. Before the brake application, the brake system has the following conditions: the pressure P_{bp} in the brake pipe is equal to the pressure reached after

the recharge; the velocity u in the brake pipe relative to the train is equal to zero (steady state, with the assumption of no leakage); the brake pipe cut-off valve is fully open, and $P_{1,3} \approx P_{bp}$. The pressure P_{eq} of the equalizing reservoir is assumed to be equal to the pressure $P_{1,3}$ of the intermediate chamber.

3. When the handle is moved towards the application position, the exhaust valve of the regulating valve is opened. Depending on the value of X_{REG}, equation (38) given by

$$\frac{\mathrm{d}P_{eq}}{\mathrm{d}t} = \frac{R_g}{V_{eq}}\left(\frac{\mathrm{d}m_{2,1}}{\mathrm{d}t} - \frac{\mathrm{d}m_{1,1}}{\mathrm{d}t}\right)$$

   can be solved along with the train's differential equations to evaluate the equalizing reservoir pressure P_{eq} using the procedure described in the case of the brake release except for the initial condition and the use of the parameters of the exhaust valve instead of those of the supply valve. The initial condition for this first-order ordinary differential equation is $P_{eq} = P_{1,3}$. Note also that in the simplified model $\dot{m}_{1,1} = 0$.

4. When the pressure P_{eq} is sufficiently low, the relay exhaust valve starts opening, letting the air flow from the brake pipe to the intermediate chamber through the brake pipe cut-off valve and then to the atmosphere through the exhaust valve. The exhaust valve's area which is a function of $D_P = P_{eq} - P_{1,3}$ is defined in equation (69) of the Appendix. If the simplified model is used, the pressure $P_{1,3}$ starts to decrease according to the equation

$$P_{1,3} = \sqrt{\left(P_{bp}^2 A_{3,3}^2 + P_a^2 A_{EX}^2\right) / \left(A_{3,3}^2 + A_{EX}^2\right)}$$

   (equation (42)), as described in the Appendix. During this process, the brake pipe cut-off valve remains fully open, that is, $A_{3,3} = C_2$ as shown in the Appendix by equation (62).

5. Using the current values of $P_{1,3}$ and $A_{3,3}$, the mass flow rate of equation (53) of the Appendix is used to determine $\dot{m}_{1,4}$ which is used in the air flow equations for the brake pipe. The pipe's pressure reduction activates the brake application mode of the CCUS (described in the companion paper[26]) and this can result in brake force application on the car wheels.

6. As the pressure $P_{1,3}$ continues to decrease, the exhaust valve closes. The brake application process, thus, ends with the relay supply valve closed and with the brake pipe cut-off valve fully open. The value of the pressure $P_{1,3}$ that determines the closing of the exhaust valve is controlled by the regulating valve in a similar manner as in the case of the brake release; as X_{REG} reaches a specified known value, P_{eq} decreases reaching its minimum value, the regulating exhaust valve closes. The minimum value of P_{eq} corresponds to a minimum value of the pressure $P_{1,1}$ that controls the closure of the relay exhaust valve.

7. When the relay exhaust valve closes, $A_{EX} = 0$, and equation (42) shows that $P_{1,3} = P_{bp}$. This means that the pressure in the intermediate chamber will assume the value of the pressure in the brake pipe.

Since the right-hand side of equation (42) also depends on the pressure $P_{1,3}$, this pressure can be obtained by solving iteratively the non-linear equation or it can be calculated using the values of P_{bp} and A_{EX} from the previous time step. The value of A_{EX} from the previous step is updated with the new value of $P_{1,3}$ as described in the Appendix.

It is important to point out that while in the model developed in this study the above-mentioned steps are valid for both the service and emergency brake application modes, in the emergency mode, the air in the brake pipe vents to the atmosphere not only through its cut-off valve but also through the emergency portion of the CCU. Furthermore, it should be mentioned that, in order to include the CCU model in the discussed cases, the time derivative of the pressures associated with the CCU have to be integrated with the locomotive valve, the brake pipe and the train's dynamic equations. These parameters are introduced and discussed in the companion paper.[26] In that companion paper, it is shown that the proposed air brake model can be used to accurately model different brake scenarios. This is demonstrated using numerical examples with different initial conditions, element numbers and track geometry. The results of these examples are validated using analytical and experimental results reported in the literature.

Summary and conclusions

The objective of this work was to integrate an air brake model that was developed using the trajectory coordinates with a non-linear dynamic model of a train. To this end, an air brake model that included three main parts: the automatic brake valve, the brake pipe and the CCU; was developed. In this study, the three main valves of a 26C automatic brake valve; the regulating valve, the relay valve and the brake cut-off valve; were considered. The general equations governing fluid behavior, including the continuity and the momentum equations, were used to develop the model of air flow in the brake pipe. Using the assumption of one-dimensional flow, one obtains two first-order partial differential equations expressed in terms of the pressure and velocity. Using the finite element method, the two partial differential equations can be converted to a system of first-order ordinary differential equations with a constant coefficient matrix. The two main scenarios that describe brake release and application were discussed and the computational algorithms for the simulation of these scenarios were presented. In a companion paper[26] the CCU and its operation during the brake release and application are discussed. The companion paper also includes numerical results that demonstrate the use of the proposed formulations and algorithms and their implementation in the computer program ATTIF developed to study the train longitudinal forces.

Funding

This research was supported by the Federal Railroad Administration, Washington, DC.

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Appendix

Calculation of the area of the regulating valve

The equations for the regulating valve presented in this paper are expressed in terms of areas that will be evaluated in this section of the Appendix. As discussed in the paper, the regulating valve areas $A_{2,3}$ and $A_{2,2}$ are functions of the valve displacement; denoted as $X_{2,1}$ (see Figure 5(c)). Assume that there are only two variable surface areas created by the relative movement of the valve with respect to its seat. One of these areas is perpendicular to the valve movement direction X, called A_x; while the other is tangent to the valve movement, called A_r. For A_x, one has

$$A_x = \pi\left(r_0^2 - r^2\left(X_{2,1}\right)\right) \tag{45}$$

where r_0 is the radius of the valve seat, and $r(X_{2,1})$ is the inner radius of the annular orifice which varies with the valve displacement. The radius $r(X_{2,1})$ can be defined as

$$r(X_{2,1}) = \frac{r_0(l - X_{2,1})}{l} \tag{46}$$

If $\alpha = 45°$, then $l = r_0$, and $r(X_{2,1})$ is given in this special case by $r(X_{2,1}) = r_0 - X_{2,1}$; and the area in this special case reduces to $A_x = A_0 - \pi(r_0 - X_{2,1})^2$. For A_r, one has

$$A_r = \pi D_I X_{2,1} \tag{47}$$

where D_I is the diameter of the inner area of the valve seat. Assuming that A_x and A_r are in series, the equivalent area A_{EQV} can be defined as

$$A_{EQV} = \frac{A_x A_r}{\sqrt{A_x^2 + A_r^2}} \tag{48}$$

A_{EQV} can be $A_{2,2}$ for the exhaust valve and $A_{2,3}$ for the supply valve.

Equilibrium of the diaphragm of the regulating valve

Because the areas $A_{2,2}$ and $A_{2,3}$ are functions of the valve displacement X_{REG}, it is important to evaluate this displacement. It can be determined by studying the equilibrium of the diaphragm of the regulating valve which is subjected to two forces; one on each of its sides. The first force is due to the equalizing pressure P_{eq} and is equal to $F_1 = -P_{eq}A_{2,1}$. The second force is due to the spring force and is defined as $F_2 = S_{2,1} = K_{2,1}X_{REG} + L_{2,1}$, where $K_{2,1}$ is the spring stiffness and $L_{2,1}$ is the spring's pre-load force. Neglecting the effect of the inertia of the diaphragm, the equilibrium condition of the diaphragm is $F_1 + F_2 = 0$, which defines X_{REG} as

$$X_{REG} = \frac{P_{eq}A_{2,1} - L_{2,1}}{K_{2,1}} \tag{49}$$

The displacement X_{REG} and the geometry of the supply and exhaust valves can be used to evaluate the areas $A_{2,2} = A_{2,2}(X_{REG})$ and $A_{2,3} = A_{2,3}(X_{REG})$.[11] The areas $A_{2,2}$ and $A_{2,3}$ can then be substituted into equation (39), which in turn is substituted into equation (38), demonstrating that the right-hand side of equation (38) depends non-linearly on the equalizing reservoir pressure P_{eq}.

The valve starts to close when X_{REG} is less than a certain value X_C, which is the maximum effective opening made by the valves obtained when the equalizing reservoir pressure P_{eq} reaches a cut-off value of P_C which is the final steady state value of P_{eq}. Therefore, as an alternative to using equation (49), one can use the following equation

$$X_{2,1} = \frac{|P_{eq} - P_C|A_{2,1}}{K_{2,1}} \tag{50}$$

with the assumption that $|P_{eq} - P_C| \leqslant K_{2,1}X_C/A_{2,1}$. In the preceding equation, X_{REG} is renamed $X_{2,1}$. Comparing equations (49) and (50), one can show that $P_C A_{2,1} = L_{2,1} + K_{2,1}(X_{REG} - X_{2,1})$.[11] Starting with an initial value for P_{eq}, equation (38) can be integrated to determine the value of the equalizing reservoir pressure P_{eq} as function of time and the position of the handle of the automatic brake valve.

Mathematical model of the relay valve

The rate of change of the outer chamber pressure $P_{1,1}$ is due to the mass flow rate $\dot{m}_{1,1}$ from the equalizing reservoir through the feedback orifice $A_{1,1}$. Keeping in mind that $V_{1,1}$ is not constant; the use of equation (34) leads to

$$\frac{dP_{1,1}}{dt} = \frac{1}{V_{1,1}}\left(R_g\Theta\frac{dm_{1,1}}{dt} - P_{1,1}\frac{dV_{1,1}}{dt}\right) \tag{51}$$

Considering the fact that $V_{1,2}$ is not always constant, the rate of change of $P_{1,2}$ in the inner chamber, due to the mass flow rate through the inner chamber orifice $A_{1,3}$ is

$$\frac{dP_{1,2}}{dt} = \frac{1}{V_{1,2}}\left(R_g\Theta\frac{dm_{1,2}}{dt} - P_{1,2}\frac{dV_{1,2}}{dt}\right) \tag{52}$$

The rate of change of the pressure $P_{1,3}$ in the intermediate constant volume $V_{1,3}$ is due to the mass flow rates $-\dot{m}_{1,2}$ through the inner chamber orifice $A_{1,3}$ (mass flow from the inner chamber); $\dot{m}_{1,3}$ through the supply valve $A_{1,6}$ or through the exhaust valve $A_{1,4}$ (plus the exhaust orifice $A_{1,5}$); and $\dot{m}_{1,4}$ through the brake pipe cut-off valve $A_{3,3}$ (mass flow from the brake pipe). Therefore, the equation for the pressure $P_{1,3}$ is

$$\frac{dP_{1,3}}{dt} = \frac{R_g\Theta}{V_{1,3}}\left(\frac{dm_{1,3}}{dt} + \frac{dm_{1,4}}{dt} - \frac{dm_{1,2}}{dt}\right) \tag{53}$$

The rates of change of the two chamber volumes depend on the diaphragm's velocity $u_{1,1} = dX_{1,1}/dt$ and are defined as

$$\frac{dV_{1,1}}{dt} = A_{1,2}u_{1,1}, \quad \frac{dV_{1,2}}{dt} = -A_{1,2}u_{1,1} \tag{54}$$

The mass flow rates that appear in the preceding equations are defined as

$$\left.\begin{aligned}
\dot{m}_{1,1} &= 0.6A_{1,1}P_{1,1}\sqrt{\frac{|r^2-1|}{R_g\Theta}\frac{|r-1|}{r-1}}, \quad r=\frac{P_{\mathrm{eq}}}{P_{1,1}}\\
\dot{m}_{1,2} &= 0.6A_{1,3}P_{1,2}\sqrt{\frac{|r^2-1|}{R_g\Theta}\frac{|r-1|}{r-1}}, \quad r=\frac{P_{1,3}}{P_{1,2}}\\
\dot{m}_{1,3} &= 0.6A_{1,6}P_{1,3}\sqrt{\frac{|r^2-1|}{R_g\Theta}\frac{|r-1|}{r-1}}, \quad r=\frac{P_{\mathrm{mr}}}{P_{1,3}}\\
&X_0 \leqslant X_{1,1} \leqslant X_{\mathrm{S}} \quad \text{(supply valve open)}\\
\dot{m}_{1,3} &= 0.6A_{\mathrm{EX}}P_{1,3}\sqrt{\frac{|r^2-1|}{R_g\Theta}\frac{|r-1|}{r-1}}, \quad r=\frac{P_{\mathrm{a}}}{P_{1,3}}\\
&\text{(otherwise)}\\
\dot{m}_{1,4} &= 0.6A_{3,3}P_{1,3}\sqrt{\frac{|r^2-1|}{R_g\Theta}\frac{|r-1|}{r-1}}, \quad r=\frac{P_{\mathrm{bp}}}{P_{1,3}}
\end{aligned}\right\} \tag{55}$$

where $A_{\mathrm{EX}} = A_{1,4}A_{1,5}/\sqrt{A_{1,4}^2 + A_{1,5}^2}$ is the equivalent area of the exhaust valve and the orifice. In order to determine the velocity $u_{1,1}$, the equation of motion of the diaphragm must be used. Using Figures 6(b) and 6(c), one can show that the diaphragm's equation of motion can be written as

$$m_{\mathrm{d1}}\frac{\mathrm{d}u_{1,1}}{\mathrm{d}t} = F_{1,1} - F_{1,2} - S_{1,1} - S_{1,2} - S_{1,3} - K_{\mathrm{D}}X_{1,1} \tag{56}$$

where $F_{1,1}$ is the force due to the action of pressure $P_{1,1}$ in the outer chamber $V_{1,1}$; $F_{1,2}$ is the force generated by pressure $P_{1,2}$ in the inner chamber $V_{1,2}$; $S_{1,1}$ is the exhaust spring force; $S_{1,2}$ is the diaphragm rod spring force; $S_{1,3}$ is the supply spring force; K_{D} is the spring constant of the diaphragm of the relay valve; and m_{d1} is the equivalent mass of all the relay valve moving parts. Clearly, m_{d1} depends on the displacement of the diaphragm since not all the parts of the relay valve are always in motion, depending on which parts are in contact with the rod. The mass m_{d1} can be written as

$$\left.\begin{aligned}
m_{\mathrm{d1}} &= m_{\mathrm{d1},1} + m_{\mathrm{d1},2} + m_{\mathrm{d1},3} && -X_{\mathrm{E}} \leqslant X_{1,1} \leqslant -X_{\mathrm{I}}\\
m_{\mathrm{d1}} &= m_{\mathrm{d1},1} && -X_{\mathrm{I}} \leqslant X_{1,1} \leqslant 0\\
m_{\mathrm{d1}} &= m_{\mathrm{d1},1} + m_{\mathrm{d1},3} && 0 \leqslant X_{1,1} \leqslant X_0\\
m_{\mathrm{d1}} &= m_{\mathrm{d1},1} + m_{\mathrm{d1},3} + m_{\mathrm{d1},4} && X_0 \leqslant X_{1,1} \leqslant X_{\mathrm{S}}
\end{aligned}\right\} \tag{57}$$

where $m_{\mathrm{d1},1}, m_{\mathrm{d1},2}, m_{\mathrm{d1},3}$ and $m_{\mathrm{d1},4}$ are the masses of, respectively, the diaphragm, the exhaust valve, the diaphragm rod and the supply valve. In the preceding system of equations, the first equation is used when the diaphragm is moving left, opening the exhaust valve; the second equation is used in the small range in which the diaphragm moves freely; the third equation is used during the gap X_0 that the rod has to travel before reaching the supply valve; and the last equation is used during the opening of the supply valve. The expressions for the forces that appear in equation (46) are presented in the reminder of this Appendix.

Forces of the relay valve's diaphragm

In this section of the Appendix, the expressions for the forces that appear in equation (56) are developed. As previouslydiscussed during the movement of the diaphragm not all the force components are active. The forces $F_{1,1} = P_{1,1}A_{1,2}$ and $F_{1,2} = P_{1,2}A_{1,2}$ are always present, whereas the force $S_{1,2}$ takes the following values in the specified ranges

$$S_{1,2} = \begin{cases} L_{1,2} + K_{1,2}X_{1,1} & 0 \leqslant X_{1,1} \leqslant X_S\\ L_{1,2} + K_{1,2}(X_{1,1} + X_{\mathrm{I}}) & -\mathrm{X}_{\mathrm{E}} \leqslant X_{1,1} \leqslant -\mathrm{X}_{\mathrm{I}}\\ 0 & \text{otherwise} \end{cases} \tag{58}$$

When the supply valve is open, $S_{1,1} = 0$, and

$$S_{1,3} = L_{1,3} + K_{1,3}(X_{1,1} - X_0) \quad \mathrm{X}_0 \leqslant X_{1,1} \leqslant X_{\mathrm{S}} \tag{59}$$

When the exhaust valve is open, $S_{1,3} = 0$, and

$$S_{1,1} = -L_{1,1} + K_{1,1}(X_{1,1} + X_{\mathrm{I}}) \quad -\mathrm{X}_{\mathrm{E}} \leqslant X_{1,1} \leqslant -X_{\mathrm{I}} \tag{60}$$

where $K_{1,1}$, $K_{1,2}$, $K_{1,3}$ are the spring constants; and $L_{1,1}$, $L_{1,2}$, $L_{1,3}$ are the spring pre-loads.

For the relay valve, there are two variable areas, the supply $A_{1,6}$ and the exhaust $A_{1,4}$, that depend on the displacement $X_{1,1}$. Note that $A_{1,6} = 0$ when $X_{1,1} < X_0$, and $A_{1,6} = \pi D_{1,6}(X_{1,1} - X_0)$ for $X_0 < X_{1,1} < X_{\mathrm{S}}$. On the other hand, $A_{1,4} = 0$ when $X_{1,1} > -X_{\mathrm{I}}$ and $A_{1,4} = -\pi D_{1,4}(X_{1,1} + X_{\mathrm{I}})$ for $-X_{\mathrm{E}} < X_{1,1} < -X_{\mathrm{I}}$. Recall that the supply valve is open only when the displacement is greater than X_0, whereas the exhaust valve is open when it is less than $-X_{\mathrm{I}}$.

Relay valve states

The relay valve has different modes of operations defined mainly by the difference between the equalizing reservoir pressure P_{eq} and the intermediate chamber pressure $P_{1,3}$. The three possible modes of operation are the *supply state*, the *intermediate state* and the *exhaust state*. Figure 8(a) shows the relationship between the relay valve states and the difference between the two pressures acting on the diaphragm surfaces, while Figure 8(b) shows the configuration of

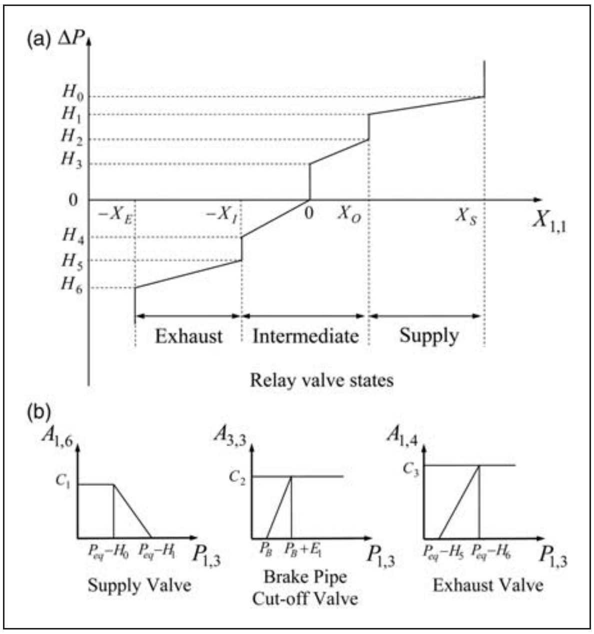

Figure 8. (a) Valve states and (b) configurations of the supply valve, brake pipe cut-off valve and exhaust valve.[11]

the relay supply valve, the brake pipe cut-off valve and the relay exhaust valve as a function of the value assumed by pressure $P_{1,3}$. By examining these figures, one can better understand the three modes of operation of the relay valve which are summarized in the remainder of this Appendix.

Supply state. The opening of the supply valve can be activated during any of the brake pipe modes (application, release/recharge, emergency). In order for the relay valve to maintain the pre-selected pressure value against the brake pipe leakage, the lapping position is considered as part of the supply state, for any of the above mentioned modes. The values of the pressure P and the area A used in equation (42) are $P = P_{\mathrm{mr}}$, and $A = A_{1,6}$. As previously mentioned, $A_{1,6}$ is a function of the valve displacement $X_{1,1}$, which is again a function of the pressure $P_{1,3}$. As shown in the previous section, $A_{1,6}$ can be determined using the following equation

$$A_{1,6} = \pi D_{1,6}\left(X_{1,1} - X_0\right) \tag{61}$$

Note that $X_{1,1}$ can be determined using equation (40). Also the brake pipe cut–off valve area $A_{3,3}$ (dependent on the valve displacement $X_{3,1}$) is function of $P_{1,3}$ (which is equal to $P_{3,1}$). Equation (44) also leads to

$$A_{3,3} = \begin{cases} 0 \quad (\text{closed}) & P_{1,3} \leqslant P_{\mathrm{B}},\ X_{3,1} = 0 \\ \pi D_{3,3} A_{3,2} \dfrac{P_{3,1} - P_{\mathrm{B}}}{K_{3,1}} & P_{\mathrm{B}} \leqslant P_{1,3} < P_{\mathrm{B}} + \varepsilon, \\ & X_{3,1} = A_{3,2} \dfrac{P_{3,1} - P_{\mathrm{B}}}{K_{3,1}} \\ \pi D_{3,3} X_{\mathrm{B}} = C_2 & P_{1,3} \geqslant P_{\mathrm{B}} + \varepsilon,\ X_{3,1} = X_{\mathrm{B}} \end{cases} \tag{62}$$

Substituting the obtained values for P and A, equation (42) can in general be written in the following form

$$f = \sum_{n=0}^{4} \beta_n \left(P_{1,3}\right)^n = 0 \tag{63}$$

This non-linear equation can be solved iteratively using a Newton–Raphson algorithm to determine the pressure $P_{1,3}$. It is important to point out that f is not a well behaved function because of the non-linearity of the coefficients $\beta_0, \beta_1, \beta_2, \beta_3$ and β_4; these coefficients are non-linear functions of $P_{1,3}$ and P_{eq}, and have different values in different regions.[11]

Figure 9 can be used to better understand the different regions and the problems associated with the non-linearity of equation (63). By increasing the pressure $P_{1,3}$, first the supply valve fully opens, while the brake pipe cut-off starts opening. This defines Region I. When $P_{1,3}$ reaches the threshold value, the cut-off valve is held fully open, defining Region II. In Region III, $P_{1,3}$ is high enough to overcome all the other forces in the relay valve, and as a consequence, the supply valve starts closing. Region IV begins when the supply valve is completely closed. Before providing more details about these regions, the following variables are introduced

$$\left.\begin{array}{ll} \alpha_1 = -\pi D_{1,6} B_0, & \alpha_2 = \pi D_{3,3} B_0\left(P_{\mathrm{eq}} - H_1\right) \\ \alpha_3 = \pi D_{3,3} \dfrac{A_{3,2}}{K_{3,1}}, & \alpha_4 = -\pi D_{3,3} \dfrac{A_{3,1}}{K_{3,1}} P_{\mathrm{B}} \end{array}\right\} \tag{64}$$

For Region I, one has $P_{\mathrm{B}} \leqslant P_{1,3} < P_{\mathrm{B}} + \varepsilon$ (the brake pipe cut-off valve moves), and $H_0 \leqslant D_{\mathrm{P}}$ (supply valve is fully open). The values of A, $A_{3,3}$ and β_i to be used in equation (63) are as follows

$$\left.\begin{array}{l} A = A_{1,6} = C_1,\ \ A_{3,3} = \pi D_{3,3} A_{3,2} \dfrac{P_{3,1} - P_{\mathrm{B}}}{K_{3,1}} = \alpha_3 P_{3,1} + \alpha_4, \\ \beta_0 = -\alpha_4^2 P_{\mathrm{bp}}^2 - C_1^2 P_{\mathrm{mr}}^2 \\ \beta_1 = -2\alpha_3\alpha_4 P_{\mathrm{bp}},\ \ \beta_2 = C_1^2 + \alpha_4^2 - \alpha_3 P_{\mathrm{bp}}^2, \\ \beta_3 = 2\alpha_3\alpha_4,\ \ \beta_4 = \alpha_3^2 \end{array}\right\} \tag{65}$$

For Region II, one has $P_{1,3} \geqslant P_{\mathrm{B}} + \varepsilon$ (brake pipe cut-off valve fully open), and $H_0 \leqslant D_{\mathrm{P}}$ (supply valve fully open). The values of A, $A_{3,3}$ and β_i to be used in equation (63) are as follows

$$\left.\begin{array}{l} A = A_{1,6} = C_1, \quad A_{3,3} = C_2 = \pi D_{3,3} X_{\mathrm{B}} \\ \beta_0 = -C_1^2 P_{\mathrm{mr}}^2 - C_2^2 P_{\mathrm{bp}}^2, \quad \beta_2 = C_1^2 + C_2^2, \\ \beta_1 = \beta_3 = \beta_4 = 0 \end{array}\right\} \tag{66}$$

For Region III, one has, $P_{1,3} \geqslant P_{\mathrm{B}} + \varepsilon$ (brake pipe cut-off valve fully open), and $H_1 < D_{\mathrm{P}} < H_0$ (supply

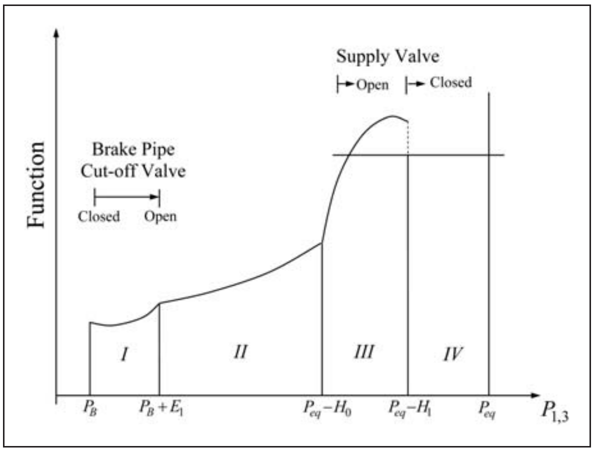

Figure 9. Supply state regions.[11]

valve moves). The values of A, $A_{3,3}$ and β_i to be used in equation (63) are as follows:

$$\left.\begin{aligned} &A = A_{1,6} = \pi D_{1,6} B_0 (D_P - H_1) = \alpha_1 P_{1,3} + \alpha_2, \\ &\quad A_{3,3} = C_2 \\ &\beta_0 = -\alpha_2^2 P_{\mathrm{mr}}^2 - C_2^2 P_{\mathrm{bp}}^2, \quad \beta_1 = -2\alpha_1\alpha_2 P_{\mathrm{mr}}^2, \\ &\beta_2 = C_2^2 + \alpha_2^2 - \alpha_1^2 P_{\mathrm{mr}}^2 \ \ \beta_3 = 2\alpha_1\alpha_2, \quad \beta_4 = \alpha_1^2 \end{aligned}\right\} \tag{67}$$

For Region IV, $P_{1,3} \geqslant P_{\mathrm{B}} + \varepsilon$ (brake pipe cut-off valve fully open), and $D_{\mathrm{P}} \leqslant H_1$ (supply valve closed). Furthermore, the exhaust valve is closed, and therefore, the function f is equal to zero.

Region V, represents another scenario, which is not shown in Figure 9. In this region, the system may operate such that both the supply and the brake pipe cut-off valves are moving. This happens if $P_{\mathrm{B}} + \varepsilon$ is high, or if H_0 is low; with values that depend on the valve construction, design and on the spring pre-loads. In this region, $P_{\mathrm{B}} \leqslant P_{1,3} < P_{\mathrm{B}} + \varepsilon$, and $H_1 < D_{\mathrm{P}} < H_0$. The values of A, $A_{3,3}$ and β_i to be used in equation (63) are as follows

$$\left.\begin{aligned} &A = A_{1,6} = \alpha_1 P_{1,3} + \alpha_2, \quad A_{3,3} = \alpha_3 P_{3,1} + \alpha_4, \\ &\quad \beta_0 = -\alpha_2^2 P_{\mathrm{mr}}^2 - \alpha_4^2 P_{\mathrm{bp}}^2 \\ &\beta_1 = -2\alpha_1\alpha_2 P_{\mathrm{mr}}^2 - 2\alpha_3\alpha_4 P_{\mathrm{bp}}^2, \quad \beta_2 = \alpha_2^2 + \alpha_4^2 - \alpha_1^2 P_{\mathrm{mr}}^2 \\ &-\alpha_3^2 P_{\mathrm{bp}}^2 \\ &\beta_3 = 2\alpha_1\alpha_2 + 2\alpha_3\alpha_4, \quad \beta_4 = \alpha_1^2 + \alpha_3^2 \end{aligned}\right\} \tag{68}$$

By using the appropriate parameters for each region, equation (63) can be solved numerically to determine the pressure $P_{1,3}$.

Intermediate state. This state can be activated during any of the brake pipe modes (application, release/recharge) and may include the lapping position provided that there is no leakage in the entire air brake system. During this state, both the supply and the exhaust valve are completely closed, that is, $A_{1,6} = A_{1,4} = A = 0$. It follows that $P_{1,3} = P_{\mathrm{bp}}$.

Exhaust state. This state can only be activated during the brake application mode. As previously shown, the equivalent exhaust area is $A_{\mathrm{EX}} = A_{1,4}A_{1,5}/\sqrt{A_{1,4}^2 + A_{1,5}^2}$, where $A_{1,4}$ is a function of $P_{1,3}$. This functional relationship is clear from the following equation

$$A_{1,4} = \begin{cases} 0 \quad (\text{closed}) & D_{\mathrm{P}} \geqslant H_5, \ X_{1,1} \geqslant -X_{\mathrm{I}} \\ -\pi D_{1,4}\left(X_{1,1} + X_{\mathrm{I}}\right) & H_6 < D_{\mathrm{P}} < H_5, \\ & -X_{\mathrm{E}} < X_{1,1} < -X_{\mathrm{I}} \\ \pi D_{1,4}(X_{\mathrm{E}} - X_{\mathrm{I}}) = C_3 & D_{\mathrm{P}} \leqslant H_6, \ X_{1,1} = -X_{\mathrm{E}} \end{cases} \tag{69}$$

During this state the brake pipe cut-off valve is always fully open. Due to the fact that $A_{1,5}$ is very small as compared with the brake pipe cut-off area, $P_{1,3}$ is very close to the value of P_{bp}; instead of solving equation (42) iteratively using the values presented in the preceding equation, one may use the values from the previous time step to calculate $P_{1,3}$, that is,

$$P_{1,3}^j = \left(\sqrt{\left(P_{\mathrm{bp}}^2 C_2^2 + P_{\mathrm{a}}^2 A_{\mathrm{EX}}^2\right)/\left(C_2^2 + A_{\mathrm{EX}}^2\right)}\right)^{j-1},$$

where subscript j refers to the current time step, while subscript $(j-1)$ refers to the previous time step.