INVESTIGACIÓN OPERACIONAL

PROMETHEE: A NEW APPROACH THROUGH FUZZY MATHEMATICAL PROGRAMMING

Resumen / *Abstract*
Los métodos multicriterios PROMETHEE se basan en las evaluaciones borrosas entre los diferentes pares de alternativas para cada criterio. PROMETHEE II asocia un número a cada acción, y la indiferencia entre dos alternativas solo ocurre cuando los flujos correspondientes son estrictamente iguales. PROMETHEE III asocia a cada acción un intervalo y dos acciones son consideradas indiferentes cuando ellas están muy cerca entre sí. PROMETHEE V aplica la Programación Lineal Entera para seleccionar el mejor subconjunto de alternativas. El objetivo es maximizar la suma de PROMETHEE II, sujeto a un conjunto de restricciones que normalmente incluyen alguna restricción financiera. En el presente trabajo se ha considerado para que el modelo sea más realista, que algunos restricciones sean suaves y que algunos coeficientes se estimen por los números borrosos. Se aplica la Programación Lineal en Enteros Borrosa, utilizando la suma de los resultados de PROMETHEE III como función de objetivo. La indiferencia introducida por PROMETHEE III permite encontrar el subconjunto de alternativas no superior y verificar la restricción suavizada. Se ilustra el método propuesto a través de un ejemplo usado en el PROMETHE V original, se comparan los dos procedimientos.

PROMETHEE multicriteria methods are all based on fuzzy evaluations of the differences between pairs of alternatives for each criterion. PROMETHEE II associates a crisp number to each action, and indifference between two alternatives only occur when the corresponding flows are strictly equal. PROMETHEE III associates to each action an interval and two actions are considered indifferent when they are very close to each other. PROMETHEE V applies Integer Linear Programming in order to select the best subset of alternatives. The objective is maximization of the sum of PROMETHEE II scorings, subject to a set of constraints, which usually include some financial constraint. In order to make the model more realistic, in this paper we consider that some constraints are soft and that some coefficients are estimated by fuzzy numbers. We apply Fuzzy Integer Linear Programming, using the sum of PROMETHEE III scorings as objective function. The indifference introduced by PROMETHEE III allows us to find the subset of not outranked alternatives that best verify the soft constraint. We illustrate our method solving the example used in original PROMETHE V presentation and we compare the two procedures.

Palabras clave / *Key words*
PROMETHEE, métodos de superioridad, números borrosos, programación matemática borrosa, presupuesto, programación lineal entera

PROMETHEE, outranking, method, fuzzy number, fuzzy mathematical, programming, capital budgeting, integer linear programming

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INTRODUCTION

The family of PROMETHEE multicriteria methods was made up by Brans *et al.*[1] in its original presentation. The methods are based on a fuzzy outranking relationship comparing all pairs of alternatives for each criterion by means of six different functions. PROMETHEE I constructs a partial preorder and PROMETHEE II develops a complete one. In PROMETHEE I and II indifference between two actions only occur when the corresponding flows are strictly equal. Nevertheless as Brans et al. say: "due to the continuous character of the generalized criteria, it may happen that for two actions a and b the flows are very close to each other. It seems, in this case, to consider there is indifference between a and b", In this way PROMETHEE III associates to each action an interval, and defines a complete non transitive interval order wich introduces indifference between two close actions. PROMETHEE V[2] uses PROMETHEE II scorings and Integer Linear Programming in order to select the best alternative subject to a set of crisp constraints, which usually include some financial constraint. Furthermore, the very nature of the constraints usually imposed in capital budgeting problems suggests the convenience of considering many of them as soft restrictions, i.e., targets rather than constraints. For instance, it is not uncommon to have some degree of flexibility to relax budgetary or, more generally, input limitations. Consider also constraints reflecting profit, sales or other output targets: it is difficult to set realistic situations where limits are not flexible. It is also natural to state in soft terms restrictions that reflect strategic objectives, such as those limiting or promoting some products, geographic areas or business centres. Our proposal in this paper is to formalize this sort of constraints using flexible equalities or inequalities. These constraints make use of a tolerance margin, so performance below it is unacceptable, performance beyond the margin is completely satisfactory, and performance within the margin is partially satisfactory (and, for simplicity, satisfaction is assumed to increase linearly). Coefficients may also be fuzzy numbers, since both evaluations of alternatives (e.g., costs or benefits from each alternative) and limits (e.g., capital budget, sales targets) are frequently subject to some degree of uncertainty. We use PROMETHEE III scoring and Fuzzy Integer Linear Programming in order to select the best alternative subject to a set of flexible constraints where some parameters can be fuzzy numbers.

We illustrate our method solving the example used in original PROMETHE V presentation and we compare the two procedures.

PROMETHEE III AND FUZZY MATHEMATICAL PROGRAMMING

Given a finite set A of n possible alternatives $\{A_1, A_2, ..., A_n\}$, which are evaluated on k criteria, PROMETHEE methods are based on a fuzzy outranking relationship: comparison of each pair of alternatives for each criterion is not necessary made in terms of a binary statement about the superiority of one alternative, but it is possible to grade the superiority (in the 0-1 interval). Preference of alternative a on alternative *b* regarding criterion *j*, *Pj*(*a*, *b*), is a function of the difference between their values (distance):

$$d = f_j(a) - f_j(b) \qquad ...(1)$$

Six different functional forms have been proposed to evaluate this distance. Some of them make use of an indifference threshold (*q*, minimum significant difference) and/or a preference threshold (*p*, completely significant difference).[2]

The preference indexes obtained evaluating the superiority of alternative a on b for the different criteria are aggregated using the relative weights of criteria (*wj*) into the global preference index of a on b. The aggregation of the indexes of preference of an alternative to all others is its positive outranking flow or leaving flow, which evaluates its outranking character, while the aggregation of the indexes of preference of all the other alternatives compared to the one considered (negative or entering flow) represents its outranked character. The difference between these two flows quantifies the relative interest of an alternative

$$\Pi(a,b) = \sum_{j=1}^{n} w_j P_j(a,b) \qquad \left(\sum_{j=1}^{m} w_j = 1\right)$$

$$\left.\begin{array}{l} \phi^+(a) = \sum_{b \in A} \Pi(a,b) \\ \phi^-(a) = \sum_{b \in A} \Pi(b,a) \end{array}\right\} \rightarrow \phi(a) = \phi^+(a) - \phi^-(a) \qquad ...(2)$$

While in PROMETHEE I and II scorings are crispy and any small difference in the flows of alternatives is regarded significant PROMETHEE III net scorings are intervals $[\underline{a}, \overline{a}]$ and it defines a complete interval order (P^{III}, I^{III}) as follows:

$$\begin{cases} a \; P^{III} \; b \quad (a \text{ outtranks } b) \quad \text{iff} \quad \underline{a} > \overline{b} \\ a \; I^{III} \; b \quad (a \text{ is indifferent to } b) \quad \text{iff} \quad \underline{a} \le \overline{b} \text{ and } \underline{b} \le \overline{a} \end{cases} \qquad ...(3)$$

We will say that an alternative a is a "not outranked alterna
if there is no other alternative that outranks a.

The interval $[\underline{a}, \overline{a}]$ is given by

$$\underline{a} = \bar{\phi}(a) - \alpha\sigma_a$$
$$\overline{a} = \bar{\phi}(a) + \alpha\sigma_a$$

$$\bar{\phi}(a) = \frac{1}{m-1} \sum_{b \in A} [\,?\,(a,b) - \,?\,(b,a)] = \frac{\phi(a)}{m-1} \qquad ...(4)$$
$$s_a^{\,2} = \frac{1}{m-1} \sum_{b \in A} [\,?\,(a,b) - \,?\,(b,a) - \bar{\phi}(a)]^2$$

Therefore, the length of the interval is proportional to the variability of the scorings obtained when the alternative considered is compared to each of the others. The parameter α usually takes the value 0,15.

In real world applications the set A of possible alternatives have additional constraints. Our proposal in this paper is to formalise this sort of constraints using flexible equalities or inequalities. These make use of a tolerance margin, so performance below it is unacceptable, performance beyond the margin is completely satisfactory, and performance within the margin is partially satisfactory (and, for simplicity, satisfaction is assumed to increase linearly). Coefficients may also be fuzzy numbers, since both evaluations of alternatives (e.g., costs or benefits from each alternative) and limits (e.g., capital budget, sales targets) are frequently subject to some degree of uncertainty. The application of the model to the problem used in original PROMETHEE V presentation will produce two clear examples of flexible constraints. But, as it will also show, of course not all constraints share this nature. Some others, such as those reflecting legal restrictions, may be considered hard limits.

We express the model that selects the best subset of alternatives using PROMETHEE III net outranking flows in the objective function, subject to a set of flexible or hard constraints with fuzzy coefficients as follows:

$$\text{Max } [\underline{z}, \overline{z}] = \sum_{i=1}^{n} [\underline{b}_i, \overline{b}_i] \; x_i$$

Subjet to:

$$\sum_{i=1}^{m} \tilde{\alpha}_{ri} \cdot x_i \sim \beta_r , \quad r = 1,..., s \qquad ...(5)$$

$$x_i \in \{0,1\} \quad i=1,2,...,n$$

where: holds for flexible or hard inequalities or equalities, $\tilde{\alpha}_{ri}$ holds for crisp or fuzzy numbers and

$$x_i = \begin{cases} 1 & \text{if } A_i \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

The set of feasible subsets of alternatives X_U is composed by those that verify the hard constraints and, in case of flexible constraints, the tolerance threshold. The coefficients of the objective function are the interval net dominance flows as defined in (4).

Let be two feasible subsets of alternatives $B, D \in X_U$, whose objective values (see expression (5) are the following intervals: $[\underline{Z}_B, \overline{Z}_B], [\underline{Z}_D, \overline{Z}_D]$. According with (3)

$$\begin{cases} B \; P^{III} D & (B \text{ outranks } D) \text{ iff } \underline{Z}_B > \overline{Z}_D \\ B \; I^{III} D & (B \text{ is indifferent to } D) \text{ iff } \underline{Z}_B \le \overline{Z}_D \text{ and } \underline{Z}_D \le \overline{Z}_B \end{cases} \qquad ...(6)$$

Proposition 1: In model (5), let be A* the subset (or subsets) of feasible alternatives with the maximum value for the lower bound $\underline{Z}$ to the interval objective value, then a feasible subset of alternatives B is not outranked if and only if B is indifferent to A*.

Proof: it is obvious from expression (6).

Taking in mind the above considerations, the procedure of our method is as follows:

- First step: through a crisp linear programming problem, we find the subset of not outranked alternatives (according PROMETHEE III) among those that verify the constraints (the tolerance margins in case of flexible constraint).

- Second step: between the aforementioned subset of not outranked alternatives we choose those that best verify the flexible constraints.

EXAMPLE

The problem selected to illustrate our new method has been the one used in Brans and Mareschal[1] for the original presentation of PROMETHEE V. It deals with the choice of some distribution centres of a firm in Belgium among 12 alternatives that we call A_i (i = 1,2,…12): 2 in the area of Antwerp (A_1,A_2), 3 in the area of Bruges (A_3,A_4,A_5), 4 in the area of Brussels (A_6, A_7,A_8,A_9) and 3 in the area of Namur (A_{10}, A_{11},A_{12}). The 12 sites are evaluated through 5 criteria: C_1 (construction cost), C_2 (number of potential costumer in the area), C_3 (number of parking places on the site), C_4 (access to the road network) and C_5 (numbers of competitors in the area), (for more detail see.[1]

The final decision is subject to the following sets of constraints:

- 1-2) Maximum and minimum number of total sites: 9 and 5.
- 3) Minimum global return: 4000 (we will suppose that this constraint is flexible and, besides, that the expected return in each site is estimated by a fuzzy number)
- 4) Minimum number of employees in Antwerp and Bruges (to get governmental incentives): 200
- 5) Wages paid in the area of Brussels may not exceed those paid in the 3 other area (we will suppose that this constraints is flexible)
- 6-9) Limits to the number of sites in each area: Antwerp only 1 site. Bruges 2 or less than 2 sites. Brussels 2 or more than 2 sites. Namur 1 or more than 1 site.
- 10-12) Due to proximity: A3 and A4, A7 and A9, A11 and A12, may not be selected together.
- 13-16) Availability of qualified manpower in each area: Antwerp 300, Bruges 200, Brussels 500, Namur, 150

Applying PROMETHEE III to the five criteria C_i (i =1,…5), with the same basic data than used by Brans and Mareschal, the interval outranking flows, according with eqs. (4), are the following

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
$\underline{b}_i$	-0,202	-0,109	0,282	0,323	0,314	-0,365	-0,417	-0,325	-0,229	0,003	0,122	0,113
$\overline{b}_i$	-0,166	-0,075	0,371	0,384	0,372	-0,307	-0,313	-0,224	-0,113	0,119	0,232	0,208

Moreover, as we have said before, we suppose that some constraints of the model proposed by Brans el al. are flexible. Specifically the global return should be expressed in the following semantic way "the global return should be essentially greater than 4000". Such linguistic expression can be modelled by a fuzzy set whose membership function represents the degree in wich the constraint is attained. In this case we consider that returns below 3500 are not allowed and that returns above 5000 are fully satisfactory (satisfaction degree=1). For simplicity, we assume that the satisfaction degree increases linearly between these two values (see fig. 1). Moreover the expected annual return of each site is an uncertain quantity, and we assume that they can be represented by symmetric triangular fuzzy numbers whose support is ±15 % of the crisp quantities estimated by Brans *et al*.

Then, the return obtained for each action $a=(x_1,x_2,\dots,x_n)$ is a fuzzy number too:

$$\tilde{R}(a)=(R_1(a),R_2(a),R_3(a))=\widetilde{416}x_1+\widetilde{645}x_2+\widetilde{76}x_3+$$
$$+\widetilde{226}x_4+\widetilde{275}x_5+\widetilde{822}x_6++\widetilde{1026}x_7+\widetilde{692}x_8+\widetilde{601}x_9+$$
$$+\ \widetilde{464}x_{10}+\widetilde{516}x_{11}+\widetilde{602}x_{12}$$

where we will use standard fuzzy arithmetic to operate with fuzzy numbers.[3]

Following Rommelfanger[4] we can replace the fuzzy constraint by (see figure 1).

$$\tilde{R}(a)\gtrsim 4000 \Leftrightarrow \begin{cases} R_1(a)\geq 3500 \\ a_1=m_{\tilde{B}}(R_2(a))\to \max \end{cases}$$

We suppose that the fifth constrain is flexible too: "the total wages paid in the area of Brussels should not exceed those paid in the 3 other areas". We will set the tolerance threshold on 50, that is to say, wages in Brussels bigger than wages in the other areas in 50 are no allowed and the satisfaction degree increases until the Brussels's wages are equal or minor than 50% of total wages.

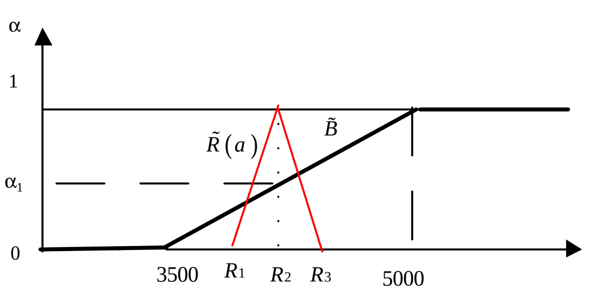

Fig. 1

RESOLUTION PROCEDURE

First step: Through a crisp linear programming problem, among the subsets of feasible alternatives we find the no outranked ones (according PROMETHEE III). Therefore, according to proposition 1, we look for the actions with the highe lower bound $\underline{Z}$ of the interval objective value:

$$\text{Max } \underline{Z} = -0{,}202x_1 - 0{,}109x_2 + 0{,}282x_3 + 0{,}323x_4 + 0{,}314x_5 -$$
$$- 0{,}365x_6 - 0{,}417x_7 - 0{,}325x_8 - 0{,}229x_9 + 0{,}003x_{10} +$$
$$+ 0{,}122x_{11} + 0{,}113x_{12}$$

Subject to

1-2)
$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}+x_{12}\geq 5$
$x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}+x_{11}+x_{12}\geq 9$

3)
$353{,}6x_1 + 549{,}1x_2 + 64{,}6x_3 + 192{,}1x_4 + 233{,}75x_5 + 698{,}7x_6 +$
$+ 872{,}1x_7 + 588{,}2x_8 + 510{,}85x_9 + 394{,}4x_{10} + 438{,}6x_{11}+$
$+ 511{,}7x_{12} \geq 3500$

4)
$118x_1 + 130x_2 + 85x_3 + 61x_4 + 52x_5 \geq 200$

5)
$63x_1 + 62x_2 + 31x_3 + 26x_4 + 37x_5 + 38x_{10} + 42x_{11} + 28x_{12} -$
$- 84x_6 - 78x_7 - 73x_8 - 69x_9 \geq -5($

6-9)
$x1 + x_2 = 1$
$x_3 + x_4 + x_5 \geq 2$
$x_6 + x_7 + x_8 + x_9 \geq 2$
$x_{10} + x_{11} + x_{12} \geq 1$

10-12)
$x_3 + x_4 = 1$
$x_7 + x_9 = 1$
$x_{11} + x_{12} = 1$

13-16)
$118x_1 + 130x_2 \geq 300$
$85x_3 + 61x_4 + 52x_5 \geq 200$
$152x_6 + 180x_7 + 130x_8 + 151x_9 \geq 500$
$66x_{10} + 76x_{11} + 50x_{12} \geq 150$
$x_i = 0$ or $1,\ i = 1,2,\dots,12$

...(7

The optimal solution is A*: $x_i=\begin{cases}1 & i=2,4,5,6,8,9,10,11\\ 0 & \text{otherwise}\end{cases}$

Whose interval objective value is

$$[\underline{Z}_{A^*},\overline{Z}_{A^*}]=[-0{,}266,0{,}388]$$

According to proposition 1, any other action D that verifies $-0{,}266\leq\overline{Z}_D$ and $\underline{Z}_D\leq 0{,}388$ is indifferent to A* and is no outranked by another feasible action.

Second step: Among all not outranked sets of alternatives we look for those that best verify the fuzzy constraints. With his purpose, following Zimmermann,[4] we solve the following nodel:

Max
Subject to

$$-0,166x_1 - 0,075x_2 + 0,371x_3 + 0,384x4 + 0,372x_5 - 0,307x_6 - 0,313x_7 - 0,224x_8 - 0,113x_9 + 0,119x_{10} + 0,232x_{11} + 0,208x_{12} \geq -0,266$$

$$-0,202x_1 - 0,109x_2 + 0,282x_3 + 0,323x_4 + 0,314x_5 - 0,365x_6 - 0,417x_7 - 0,325x_8 - 0,229x_9 + 0,003x_{10} + 0,122x_{11} + 0,113x_{12} \geq 0,388$$

$$416x_1 + 645x_2 + 76x_3 + 226x_4 + 275x_5 + 822x_6 + 1026x_7 + 692x_8 + 601x_9 + 464x_{10} + 516x_{11} + 602x_{12} - 1500 \geq 3500$$

$$53x_1 + 62x_2 + 31x_3 + 26x_4 + 37x_5 + 38x_{10} + 42x_{11} + 28x_{12} - 84x_6 - 78x_7 - 73x_8 - 69x_9 - 50 \geq -50$$

+ Constraints of model (7)

$0 \leq a \leq 1$

The optimal action is A**: $x_i = \begin{cases} 1 & i = 2,4,5,6,8,9,10,11 \\ 0 & \text{otherwise} \end{cases}$

whose interval objective value, in model (5), is: $[-0,588, 0,084]$

Let us compare our solution A** with the solution obtained by Brans *et al*.[1] through PROMETHEE V, i.e., with

$$x_i = \begin{cases} 1 & i = 2,4,5,6,7,10,12 \\ 0 & \text{otherwise} \end{cases}$$

If we apply this solution to our model (5) we obtain the following interval objective value: $[-0,138, 0,396]$, and according with PROMETHEE III scoring (see eq. (3)), it is indifferent to the solution A** obtained by our method. Regarding the flexible constraints, our solution produces a total annual return of about 4,297 and the wages paid in Brussels exceed those paid in the other 3 areas in 24. The PROMETHEE V solution produces a total annual return of about 4,060 and the total wages paid in Brussels are minor than those paid in the other 3 areas in 29. Therefore, our solution produces a much better return than the PROMETHEE V solution, but in exchange for the violation of the crisp PROMETHEE V constraint corresponding to the balance between the wages in the different areas. Our method can be used in an interactive way: if the Decision Maker (DM) does not like the solution he/she can change the tolerance threshold of flexible constraints.

CONCLUSIONS

We have developed a new approach, based on PROMETHEE V, to select the most suitable action within a finite set of possible alternatives. PROMETHEE V uses PROMETHE II and crisp Integer Linear Programming. We use PROMETHEE III and Fuzzy Integer Linear Programming, which, in our opinion, produces a more realistic model, since, usually, some constraints imposed in a real word problem should be considered soft. It is very common to work in an uncertainty atmosphere, being more realistic, and comfortable to the DM, to estimate some coefficients by fuzzy numbers than by a crisp one. Our method combines the PROMETHEE III scoring, that introduces indifference between two close actions, and the fuzzy logic, supplying a interactive tool to the Decision Maker in order to decide between a set of very closed actions, finding those that best verify the soft constraints. ◘

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